Diffusion of waves in a layer with a rough interface

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Sound trapped between the rough boundaries of a statistically homogeneous layer can exhibit diffusive behavior which influences the reverberation of a pulsed signal. If the interfaces of the layer can transmit sound, then eventually sound within the layer will be lost and there will be no diffusion. Nevertheless, if the transmission to the exterior of the layer is weak, there will be a remnant of diffusion. This paper examines the description of this kind of quasidiffusive behavior for sound which impinges on the rough interface of such a layer from outside the layer. As in the case of diffuse light scattering by particles in suspension, the diffusion constant is renormalized according to the delay required to build up resonant energy in the layer. In addition, when there is a density contrast between the interior and exterior of the layer, or when there is dispersion, the diffusion constant has another correction associated with energy flux within the layer.

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I. INTRODUCTION

Consider a fluid layer bounded below by an acoustically impenetrable flat surface and separated from a fluid half-space above by a rough interface, as indicated in Fig. 1. The sound speed and density of the layer differ from the sound speed and density of the half-space. If a point source of sound is located in the half-space near the interface, some of the evanescent waves associated with the source may coincide with what would be normal modes in the layer if the interface were flat and not rough. How do these waves propagate and scatter? In a previous paper [1] it was shown that reverberation from a pulsed source within a layered waveguide with rough boundaries behaves diffusively, decaying slowly as $1/t$ in the absence of attenuation. There are apparently two reasons why this diffusion of process needs to be treated with more caution, however. The first involves the role of the time-reversed sequences of scattering events that lead to diffusion. These time-reversed sequences lead to enhanced backscattering [2], on the one hand, and to infinities in the computation of the diffusion constant in two dimensions on the oher hand [3]. For scattering by layers this effect was considered by Sánchez-Gil *et al.* [4]. It will not be treated here. The second reason that diffusion requires more careful treatment results from resonant scattering. In the case of scattering by small objects, Kogan and Kaveh [5] pointed out that if the Boltzmann equation that governs the scattering is modified to account approximately for the time that light spends rattling around within the scatterers, then the diffusion constant will be reduced. In resonant scattering, waves can spend a sufficient time within the scatterers to alter the diffusion constant significantly. The aim of the present paper is to examine this effect in scattering by a layer which supports normal modes which, when the boundary is rough, become resonant states.

The discussion given by Kogan and Kaveh [5] of the effect of resonance on diffusive light scattering is somewhat phenomenological. To put things on a firmer foundation, Barabanenkov and Ozrin considered a frequency-independent dielectric constant. In this case the effective scattering potential takes the simple form $V=\epsilon(r)\omega^2/c^2$, so that the potential is frequency dependent, although in a very simple fashion. They noted that because of this simple frequency dependence, and because V is local in position [it is of the form $V(r,r')=v(r,\omega)\delta(r-r')$], the Ward identity expressing energy conservation can be generalized to apply to fields of differing frequencies. This generalization of the Ward identity to the case of two frequencies leads to a revised diffusion coefficient. Van Tiggelen *et al.* [6], in a comment on the work of Barabanenkov and Ozrin, showed that, in fact, the renormalization of the diffusion constant, *D*, in this case is related to the potential energy of the resonant wave which is contained within the scatterers. [5]. Van Tiggelen *et al.* obtained their result by careful consideration of the expression given by Barabanenkov and Ozrin which in turn is based on the special quadratic dependence of the local ''potential'' on frequency. Though the arguments of Kogan and Kaveh seem to be perfectly general, the methods of

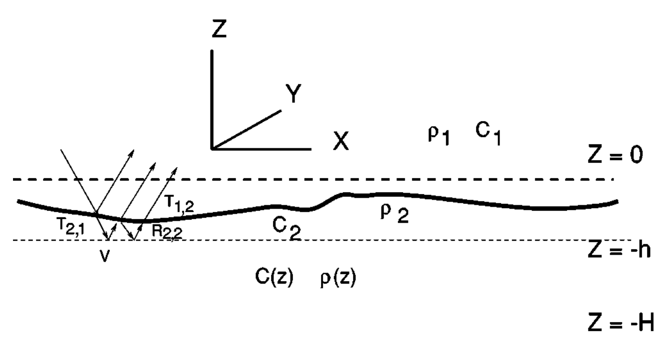

FIG. 1. Geometry of scattering by a layer. The density ρ_2 and the sound speed c_2 are the limiting values of $\rho(z)$ and $c(z)$ as the rough interface is approached from below. Waves are incident on the rough interface from above, penetrate the interface, and then scatter back and forth between the flat surface at $z=-H$ and the rough interface. The transmission amplitude from the half-space above the interface to a half-space below the interface with constant sound speed and density c_2 and ρ_2 is $T_{2,1}(\mathbf{Q}|\mathbf{K})$. Likewise, upward transmission is described by $T_{1,2}(\mathbf{Q}|\mathbf{K})$ and reflection from below by $R_{2,2}(\mathbf{Q}|\mathbf{K})$. Reflection from the imaginary surface at $z=-h$ is described by $V(\mathbf{K})$, assuming that the medium between $z=-H$ and $z=-h$ is transversely homogeneous.

Barabanenkov and Ozrin and the comment of Van Tiggelen *et al.* seem to be tied to this special model of local dielectric scattering. In the case of acoustic scattering by particles with a sound speed *and* density differing from their surroundings, their arguments do not seem to work. The situation becomes even less clear when scattering by the interface of a layer is considered. Then the role of the potential is played by the surface admittance, which is a complicated function of frequency and, more significantly, is a nonlocal function of position. What this means is that when the admittance is expresed in terms of wave vectors, it is not simply a function of the difference between incoming and outgoing wave vectors. It is this aspect of the admittance that prevents application of the methods used by Barabanenkov and Ozrin [7]. Livdan and Lisyansky [8] also considered the issue of particle scattering, and showed that, without using the Ward identity for separated frequencies, a second sort of renormalization of the diffusion coefficient is required, in addition to that considered by Barabanenkov and Ozrin. The purpose of the present paper is to show how in the case of acoustic scattering by the interface of a layer, the renormalization of the diffusion constant is tied to the potential energy of resonant states within the layer. Furthermore, the second renormalization proposed by Livdan and Lisyansky is related to the energy flux within the layer. In this way the idea of Kogan and Kaveh, that the delay due to the buildup of energy within scatterers can affect transport properties, is shown to hold in more complicated situations than simple local dielectric scattering. The main result of this work is contained in Eq. (37), with λ_0^0 given in Eq. (48), a in Eq. (61), and A in Eq. (68).

II. FORMALISM FOR INTERFACE SCATTERING

Maradudin and co-workers wrote numerous papers exploiting an admittance formalism for surface scattering [4,9,10]. The advantage of focusing on the admittance (or impedance) is that energy conservation is expressed by the simple statement that the admittance operator is Hermitian. It is the conservation of energy that leads to diffusion. Another advantage of focusing on the admittance is that reflection amplitudes are then expressed in a form that looks like the resolvent in quantum mechanics, for which there are well-developed techniques for treating multiple scattering.

If waves are incident from above on the interface, fields in the (upper) half-space can be represented by

$$\psi(\mathbf{R},z;\mathbf{K}) = \exp(i\mathbf{K}\cdot\mathbf{R})\exp(-i\beta_1(\mathbf{K})z) + \int d\mathbf{Q}\,\exp(i\mathbf{Q}\cdot\mathbf{R})\exp(+i\beta_1(\mathbf{Q})z)R(\mathbf{Q}|\mathbf{K}). \tag{1}$$

Here $\mathbf{K}$ is the horizontal projection of the wave vector of the incident plane wave. The vertical components of wave vectors are denoted by

$$\beta_i(\mathbf{K}) \equiv \sqrt{\omega^2/c_i^2 - \mathbf{K}^2}. \tag{2}$$

The sound speed above the interface is c_1, and the sound speed *just* below the interface is c_2. As will be seen, the formalism allows for depth-dependent sound speeds in the layer. If $S(\mathbf{Q}|\mathbf{K}) \equiv \sqrt{\beta_1(\mathbf{Q})}R(\mathbf{Q}|\mathbf{K})/\sqrt{\beta_1(\mathbf{K})}$, then energy conservation is expressed by

$$\Theta = S^\dagger \Theta S - i(S^\dagger \bar{\Theta} - \bar{\Theta} S), \tag{3}$$

where

$$\Theta(\mathbf{p}) = \begin{cases} 1 & \text{if } p < \omega/c_1 \\ 0 & \text{otherwise}, \end{cases} \tag{4}$$

and $\bar{\Theta} = 1 - \Theta$ [11].

This result accounts for both propagating and evanescent waves. [Note that the term beginning with $-i$ in Eq. (3) was given with the incorrect sign in Ref. [11].] On a flat surface $z=0$, just above the highest point of the interface we can use the reflection operator to construct a formal relationship between the field and its z derivative:

$$\partial_z \psi(\mathbf{Q},z|\mathbf{K})|_{z=0} = -\int d\mathbf{P}\,Y(\mathbf{Q}|\mathbf{P})\psi(\mathbf{P},0|\mathbf{K}). \tag{5}$$

(The minus sign is chosen according to the convention that Y relates the *outward* normal derivative to the field itself. At a lower boundary, the outward derivative is $-\partial_z$.) The Fourier transform convention used here will be

$$\psi(\mathbf{Q},z|\mathbf{K}) = \left(\frac{1}{2\pi}\right)\int d\mathbf{R}\,e^{-i\mathbf{Q}\cdot\mathbf{R}}\psi(\mathbf{R},z|\mathbf{K}). \tag{6}$$

The relationship between the reflection operator R and the admittance Y is

$$R = \frac{1}{(1-(i/\beta_1)Y)}(1+(i/\beta_1)Y). \tag{7}$$

The order of the β's and Y's is important. If a β is located to the left of a Y, it is to be evaluated using the horizontal wave vector of the left argument of Y, i.e., $(\beta_1 Y)(\mathbf{K}|\mathbf{Q}) = \beta_1(\mathbf{K})Y(\mathbf{K}|\mathbf{Q})$. In the special case of a flat interface at $z=0$ and a homogeneous layer of density ρ_2 and sound speed c_2, bounded below at $z=-H$ where the field vanishes, the admittance Y_0 is diagonal in wave number, and is given by

$$Y_0(\mathbf{Q}) = -\rho_1\beta_2(\mathbf{Q})\cot(\beta_2(\mathbf{Q})H)/\rho_2. \tag{8}$$

For a slightly rough interface at $z=0$, with roughness specified by $z=h(\mathbf{R})$, the surface height, the first-order deviation of the admittance from the flat interface result is

$$\Delta Y(\mathbf{Q}|\mathbf{K}) = \hat{h}(\mathbf{Q}-\mathbf{K})\left[Y_0(\mathbf{Q})(1-\rho_2/\rho_1)Y_0(\mathbf{K}) + (k_1^2 - \mathbf{Q}\cdot\mathbf{K}) - \frac{\rho_1}{\rho_2}(k_2^2 - \mathbf{Q}\cdot\mathbf{K})\right] \tag{9}$$

This result can be obtained from the perturbation results of Ivakin [12]. Note that in previous papers [1] $Z=1/Y$ was used; Maradudin used Y. Also note that ΔY is not simply a function of the wave vector difference $\mathbf{Q}-\mathbf{K}$. It is this fact that requires a slightly different approach than that used by Barabanenkov and Ozrin.

The quantity $1+R$ not only appears in the Green function above the interface, but also can be cast into the form of a surface Green's function, since

$$1+R=-2i\frac{1}{g_0^{-1}-y}\beta_1, \tag{10}$$

where

$$g_0^{-1}\equiv -i\beta_1-Y_0 \tag{11}$$

and

$$y=Y-Y_0. \tag{12}$$

Thus if the moments of

$$g=\frac{1}{g_0^{-1}-y} \tag{13}$$

can be found, so can the moments of the scattered fields exterior to the scattering layer.

Following Maradudin and co-workers [4,9,10], fluctuations of the Green's function in the upper half-space are given by

$$\Delta G(\mathbf{Q},z|\mathbf{K},z_0)=\langle G\rangle(\mathbf{Q},z,z=0)t(\mathbf{Q}|\mathbf{K})\langle G\rangle(\mathbf{K},z=0,z_0), \tag{14}$$

where

$$t=\frac{1}{1-v\langle g\rangle}v \tag{15}$$

and

$$v=y-M=y-(g_0^{-1}-\langle g\rangle^{-1}). \tag{16}$$

The self-energy M is the difference between the admittance of the mean field and the admittance of a flat interface separating the two media. This formalism was developed in Ref. [9]. Because the admittance Y is used rather than the impedance, ΔG in Eq. (14) involves the average of G rather than the derivatives of the average of G as in Ref. [1].

The second moment of the scattering operator t, $\langle tt^*\rangle$, is needed to describe intensities and correlations. To be consistent with Refs. [6,8,13], let the correlation of the surface Green's functions be given by

$$\Phi_{\mathbf{P},\mathbf{K}}(\mathbf{p}|\Omega,\omega)\,\delta(\mathbf{p}-\mathbf{k})\equiv\langle g(\mathbf{P}+\mathbf{p}/2,\mathbf{K}+\mathbf{k}/2,\Omega+\omega/2)g^*(\mathbf{P}-\mathbf{p}/2,\mathbf{K}-\mathbf{k}/2,\Omega-\omega/2)\rangle, \tag{17}$$

assuming statistical homogeneity. Following Baranbanenkov and Ozrin in Ref. [14] (but note that the signs here differ) we use the notation

$$2i\Delta G_{\mathbf{P}}(\mathbf{p}|\Omega,\omega)=\langle g\rangle(\mathbf{P}+\mathbf{p}/2,\Omega+i\epsilon+\omega/2)-\langle g\rangle^*(\mathbf{P}-\mathbf{p}/2,\Omega+i\epsilon-\omega/2), \tag{18}$$

$$2i\Delta M_{\mathbf{P}}(\mathbf{p}|\Omega,\omega)=M(\mathbf{P}+\mathbf{p}/2,\Omega+i\epsilon+\omega/2)-M^*(\mathbf{P}-\mathbf{p}/2,\Omega+i\epsilon-\omega/2), \tag{19}$$

and for the scattering operator U write

$$U_{\mathbf{P},\mathbf{P}'}(\mathbf{q}|\Omega,\omega)=\Delta G_{\mathbf{P}}(\mathbf{q}|\Omega,\omega)K_{\mathbf{P},\mathbf{P}'}(\mathbf{q}|\Omega,\omega)-\Delta M_{\mathbf{P}}(\mathbf{q}|\Omega,\omega)\,\delta_{\mathbf{P},\mathbf{P}'}, \tag{20}$$

where K is the irreducible vertex function. In these expressions we assume that Ω has a small positive imaginary part $i\epsilon$, and that the Fourier transform from frequency to time follows the convention

$$f(t)=\int\frac{d\omega}{2\pi}\,e^{-i\omega t}\hat{f}(\omega).$$

In this way, if $\hat{f}$ is analytic in the upper half complex frequency plane, $f(t)=0$ for $t<0$.

The correlation function Φ is determined from the Bethe-Salpeter equation

$$(i/2)[g_{0,\mathbf{p}+\mathbf{q}/2}^{-1}(\Omega+\omega/2)-g_{0,\mathbf{p}-\mathbf{q}/2}^{*-1}(\Omega-\omega/2)]\Phi_{\mathbf{p},\mathbf{k}}(\mathbf{q}|\Omega,\omega)-\int d\mathbf{p}'\,U_{\mathbf{p},\mathbf{p}'}(\mathbf{q}|\Omega,\omega)\Phi_{\mathbf{p}',\mathbf{k}}(\mathbf{q}|\Omega,\omega)=\Delta G_{\mathbf{p}}(\mathbf{q}|\Omega,\omega)\,\delta_{\mathbf{p},\mathbf{k}}. \tag{21}$$

Diffusion is a consequence of energy conservation [7]. In the present case energy conservation is expressed by the Ward identity [9]. Let S denote the quantity

$$S_{\mathbf{p}''}(\mathbf{k}|\Omega,\omega)=2i\int d\mathbf{p}'\,d\mathbf{p}U_{\mathbf{p},\mathbf{p}'}(\mathbf{k}|\Omega,\omega)\Phi_{\mathbf{p}',\mathbf{p}''}(\mathbf{k}|\Omega,\omega). \tag{22}$$

One form of the Ward identity states that S is also given by

$$S_{\mathbf{p}''}(\mathbf{k}|\Omega,\omega)=\int d\mathbf{p}d\mathbf{p}'\,d\mathbf{q}\langle[y^*_{\mathbf{p}-\mathbf{q}/2,\mathbf{p}'-\mathbf{q}/2}(\Omega-\omega/2)-y_{\mathbf{p}'+\mathbf{q}/2,\mathbf{p}+\mathbf{q}/2}(\Omega+\omega/2)]\times g_{\mathbf{p}+\mathbf{q}/2,\mathbf{p}''+\mathbf{k}/2}(\Omega+\omega/2)\times g^*_{\mathbf{p}'-\mathbf{q}/2,\mathbf{p}''-\mathbf{k}/2}(\Omega-\omega/2)\rangle. \tag{23}$$

This result can be demonstrated from the Bethe-Salpeter equation (21) and application of

$$g_0^{-1}g=1+yg. \tag{24}$$

When $\omega=0$ and $\mathbf{q}=0$, the Ward identity becomes

$$\int d\mathbf{p} U_{\mathbf{p},\mathbf{p}'}(0|\Omega,0)=0. \tag{25}$$

If energy is conserved, then $y(\Omega)=y^{\dagger}(\Omega)$. In the case of quantum mechanical potential scattering, where the role of y is played by a frequency-independent local potential, the admittance y takes the form, $y_{\mathbf{p},\mathbf{p}'}\rightarrow\hat{v}(\mathbf{p}-\mathbf{p}')$. This causes the expression on the right of Eq. (23) to vanish independently of ω. In the case of dielectric scattering by particles with frequency-independent dielectric constant, discussed in Refs. [6,8,14], the role of y is played by $\Omega^2/c^2\epsilon(r)$. This means that $y_{\mathbf{p}'+\mathbf{q}/2,\mathbf{p}+\mathbf{q}/2}(\Omega+\omega/2)=[(\Omega+\omega/2)^2/c^2]\hat{\epsilon}(\mathbf{p}'-\mathbf{p})$, and that

$$[y^{*}_{\mathbf{p}-\mathbf{q}/2,\mathbf{p}'-\mathbf{q}/2}(\Omega-\omega/2)-y_{\mathbf{p}'+\mathbf{q}/2,\mathbf{p}+\mathbf{q}/2}(\Omega+\omega/2)] =-(2\omega/\Omega)y_{\mathbf{p}',\mathbf{p}}(\Omega). \tag{26}$$

Hence, if $\omega=0$ and $\mathbf{k}\neq 0$, $S_{\mathbf{p}''}(\mathbf{k}|\Omega,0)=0$. It is this fact that allows for the simplifications that are described by Barabanenkov and Ozrin. For acoustic scattering when there is a density contrast, or when the dielectric function is non-local, as in the polariton problem treated by Maradudin *et al.*, Eq. (26) is not valid. In particular, in these cases, it is not true that y is a function only of the difference in wave vectors, i.e., $y_{\mathbf{p},\mathbf{p}'}\neq y(\mathbf{p}-\mathbf{p}')$.

Field fluctuations are found from the reducible vertex function τ,

$$\langle tt^{*}\rangle=\tau=K+K\Phi K, \tag{27}$$

as in Refs. [1,4]. In this paper only the long-time behavior of correlations is considered; for this purpose it is sufficient to determine the behavior of Φ for small ω and $\mathbf{q}$. Note that even if it appears that some contribution to Φ diverges as the roughness of the interface vanishes, because K vanishes as the roughness vanishes, the consequences of the divergence of Φ for τ need not be catastrophic.

III. RENORMALIZATION OF DIFFUSION

The principal results of this section are that the long-time behavior of a pulse scattered by a layer with a rough boundary is given by Eqs. (36), (37), and (38). The constant, a is a renormalization of the diffusion constant, D, and is given in Eq. (56), generally, and reexpressed in terms of modal averages in Eq. (61). The renormalization constant A is given in Eq. (68). The method of Barabanenkov and Ozrin [14] is adapted to the case of scattering by an acoustic layer with a density and sound speed which differ from the half-space above the layer.

Consider Eq. (11) for g_0^{-1}:

$$g_0^{-1}=-i\beta_1(\mathbf{Q},\Omega)-Y_0(\mathbf{Q},\Omega).$$

The flat-interface admittance Y_0 is real (as mentioned, if energy is conserved, Y will be Hermitian), but the vertical component of the wave vector, $\beta_1(\mathbf{Q},\Omega)$, can be real or imaginary. In the former case, waves are free to propagate away from the interface, in which case there is scattering from the interface by waves incident from above. In the latter case, waves decay exponentially from the interface. It is in this regime that we can look for diffusion within the layer.

One way to deal with the Bethe-Salpeter equation (21) is to follow the proceedure used by Barabanenkov and Ozrin in Refs. [7,14], and to explore the eigenfunctions and eigenvalues of the operator $H(\mathbf{q},\omega)$ given by

$$H(\mathbf{q},\omega)_{\mathbf{p},\mathbf{p}'}=(i/2)[g^{-1}_{0,\mathbf{p}+\mathbf{q}/2}(\Omega+\omega/2) -g^{*-1}_{0,\mathbf{p}-\mathbf{q}/2}(\Omega-\omega/2)]\delta_{\mathbf{p},\mathbf{p}'}-U_{\mathbf{p},\mathbf{p}'}(\mathbf{q}|\Omega,\omega), \tag{28}$$

and which appears in Eq. (21). In terms of H the Bethe-Salpeter equation becomes

$$H\Phi=\Delta G.$$

To determine the echo of a pulse long after it has encountered the interface for the first time requires examining the behavior of Φ for small frequency differences ω. Since quasidiffusive behavior is expected in this limit as a result of the pulse rattling within the scattering layer, with the wave energy tending to become uniform, the limit $\mathbf{q}\rightarrow 0$ is also relevant. If there is such rattling around it is because wave energy within the layer is nearly conserved. Therefore, H should be approximated in a way that allows energy conservation to be used explicitly. This is one reason for focusing on the admittance Y rather than on the scattering amplitudes as in Ref. [15]. The energy that is nearly conserved is associated with modes in the scattering layer which are nearly trapped there. For these reasons, consider the following division of the operator H. Let $H=H_0+\Delta H$, with

$$H^{0}_{\mathbf{p},\mathbf{p}'}(\mathbf{q},\omega)=\bar{\Theta}(\mathbf{p})(i/2)[g^{-1}_{0,\mathbf{p}+\mathbf{q}/2}(\Omega+\omega/2) -g^{*-1}_{0,\mathbf{p}-\mathbf{q}/2}(\Omega-\omega/2)]\delta_{\mathbf{p},\mathbf{p}'}-U_{\mathbf{p},\mathbf{p}'}(\mathbf{q}|\Omega,\omega) \tag{29}$$

and

$$\Delta H_{\mathbf{p},\mathbf{p}'}(\mathbf{q},\omega)=\Theta(\mathbf{p})(i/2)[g^{-1}_{0,\mathbf{p}+\mathbf{q}/2}(\Omega+\omega/2) -g^{*-1}_{0,\mathbf{p}-\mathbf{q}/2}(\Omega-\omega/2)]\delta_{\mathbf{p},\mathbf{p}'}. \tag{30}$$

In this way the entire irreducible vertex function U is contained in H^0, and the Ward identity can be used to deal with the eigenvalues of H^0. What happens outside the layer, without considering the coupling to the interior of the layer, is described by ΔH, and is to be regarded as a perturbation. To treat the small $\mathbf{q}$ and ω limit, consider the expansion of H to first order in ω and to second order in $\mathbf{q}$:

$$H^{1}_{\mathbf{p},\mathbf{p}'}(\mathbf{q},\omega)=H^{0}_{\mathbf{p},\mathbf{p}'}(\mathbf{q},\omega)-H^{0}_{\mathbf{p},\mathbf{p}'}(\mathbf{0},0) \approx\bar{\Theta}(\mathbf{p},\Omega)\left[-i\frac{\Lambda(\mathbf{p},\Omega)\Omega\omega}{c^2(\mathbf{p},\Omega)} +i\Lambda(\mathbf{p},\Omega)\mathbf{q}\cdot\mathbf{p}\right]\delta_{\mathbf{p},\mathbf{p}'}-\omega\frac{\partial U_{\mathbf{p},\mathbf{p}'}}{\partial\omega'}-\mathbf{q}\frac{\partial U_{\mathbf{p},\mathbf{p}'}}{\partial\mathbf{q}'} \tag{31}$$

and

$$H^2_{\mathbf{p},\mathbf{p}'}(\mathbf{q},\omega)=q_iq_j\left.\frac{\partial^2 H^0}{\partial q_i\partial q_j}\right|_{\mathbf{q}=0,\omega=0}. \qquad (32)$$

In Eq. (31),

$$\frac{\Lambda}{c^2}=-\frac{\partial\,\mathrm{Re}\,g_0^{-1}}{\partial\Omega^2} \qquad (33)$$

and

$$\Lambda=+\frac{\partial\,\mathrm{Re}\,g_0^{-1}}{\partial p^2}. \qquad (34)$$

For $p<\Omega/c_1,\Lambda/c^2=\partial Y_0/\partial\Omega^2>0$. For the flat-surface admittance used here, one can check that Λ/c^2 is positive. [By using a method similar to that used in Appendix A, it can be shown that $\Lambda/c^2>0$ generally. Likewise $\Lambda(\mathbf{p},\Omega)$ is also positive if g_0 depends on $\mathbf{p}$ only through β_i.]

The expression for H^2 must be taken with a grain of salt, since the nonanalyticity of $\beta_{1,2}$ requires particular care in the expansion of integrals involving H^2; the issue is that the limits of integration also depend on momentum. Furthermore, g_0^{-1} has poles as well as zeros. However, since g_0^{-1} multiplies $g=1/(g_0^{-1}-y)$ in $\Phi=\langle gg^*\rangle$, it will be assumed that the poles of g_0^{-1} cause no particular problems in the Bethe-Saltpeter equation.

The solution of the Bethe-Salpeter equation can be written in terms of the eigenfunctions and eigenvalues of H, $\phi^m_{\mathbf{p}}(\mathbf{q},\omega)$, and $\lambda^m(\mathbf{q},\omega)$. In the case of scattering by particles, when energy is conserved, H has an eigenfunction whose eigenvalue vanishes when $\omega=0$ and $\mathbf{q}=0$. Diffusion is determined by the wave vector and frequency dependence of this eigenvalue when $\mathbf{q}$ and ω are near zero (the hydrodynamic limit). In the present case of a layer coupled to a half-space, energy within the layer is not conserved. Nevertheless there will be an eigenvalue, say $\lambda^0(\mathbf{q},\omega)$, which becomes small in the the hydrodynamic limit, and which would vanish if the coupling to the half-space above the layer were to vanish. We will suppose that the behavior of Φ for long times is controlled by this eigenvalue and the corresponding eigenfunction. Following Barabanenkov and Ozrin, we write

$$\lambda^0(\mathbf{q},\omega)\approx\lambda^0_0-i\omega\Omega(1+a)\Lambda_0/c_0^2+Aq^2, \qquad (35)$$

where Λ_0 and c_0 are averages of Λ and c defined below. The time dependence of the reverberant field is then found from the Fourier transform [1],

$$\int d\mathbf{q}\int d\omega\frac{\exp[-i\omega t+i\mathbf{q}\cdot\mathbf{R}]}{\lambda^0_0-i\omega\Omega\Lambda_0(1+a)/c_0^2+Aq^2}$$
$$=\frac{2\pi^2c_0^2}{(1+a)\Omega 4Dt}\exp[-\mathbf{R}^2/(4Dt)-t/\tau], \qquad (36)$$

with

$$D=Ac_0^2/\Lambda_0(1+a)\Omega \qquad (37)$$

and

$$\tau=\Lambda_0(1+a)\Omega/(c_0^2\lambda^0_0)=A/(D\lambda^0_0). \qquad (38)$$

The remainder of this section is devoted to determining the constants a and A. Following van Tiggelen *et al.* [6], it will be shown these quantities can be expressed in terms of certain energylike integrals over the half-space $z>-H$.

From the definition of U and reciprocity of K, it follows that H has a symmetry

$$H(\mathbf{q},\omega)_{\mathbf{p},\mathbf{p}'}=H(-\mathbf{q},\omega)_{-\mathbf{p}',-\mathbf{p}}\frac{\Delta G_{-\mathbf{p}}(-\mathbf{q},\omega)}{\Delta G_{\mathbf{p}'}(\mathbf{q},\omega)}. \qquad (39)$$

We will assume that $g(-\mathbf{p},\Omega)=g(\mathbf{p},\Omega)$, so that $\Delta G_{-\mathbf{p}}(-\mathbf{q},\omega)=\Delta G_{\mathbf{p}}(\mathbf{q},\omega)$. As indicated in Ref. [13] this means that, in contrast to the usual Hamiltonian in quantum mechanics, the operator H has distinct left and right eigenfunctions. If $\phi^n_{\mathbf{p}}(\mathbf{q},\omega)$ is a right eigenfunction of $H_{\mathbf{p},\mathbf{p}'}$ with eigenvalue $\lambda^n(\mathbf{q},\omega)$, then $\phi^n_{-\mathbf{p}}(-\mathbf{q},\omega)/\Delta G_{-\mathbf{p}}(-\mathbf{q},\omega)$ is a left eigenfunction with eigenvalue $\lambda^n(-\mathbf{q},\omega)$. In fact, these two eigenvalues are equal, since the symmetry of H implies

$$\int d\mathbf{p}d\mathbf{p}'\,\phi^m_{-\mathbf{p}}(-\mathbf{q})\frac{1}{\Delta G_{-\mathbf{p}}(-\mathbf{q})}H_{\mathbf{p},\mathbf{p}'}(\mathbf{q})\,\phi^n_{\mathbf{p}}(\mathbf{q})$$
$$=\lambda^n(\mathbf{q})\int d\mathbf{p}d\mathbf{p}'\,\phi^m_{-\mathbf{p}}(-\mathbf{q})\frac{1}{\Delta G_{-\mathbf{p}}(-\mathbf{q})}\phi^n_{\mathbf{p}'}(\mathbf{q}) \qquad (40)$$
$$=\lambda^m(-\mathbf{q})\int d\mathbf{p}d\mathbf{p}'\,\phi^m_{-\mathbf{p}}(-\mathbf{q})\frac{1}{\Delta G_{-\mathbf{p}}(-\mathbf{q})}\phi^n_{\mathbf{p}}(\mathbf{q}). \qquad (41)$$

The ω dependence has been left implicit here. As usual, if the eigenvalues are distinct, the eigenfunctions are orthogonal with weight $1/\Delta G$. The eigenfunctions can be normalized so that

$$\int d\mathbf{p}d\mathbf{p}'\,\phi^m_{-\mathbf{p}}(-\mathbf{q})\frac{1}{\Delta G_{-\mathbf{p}}(-\mathbf{q})}\phi^n_{\mathbf{p}}(\mathbf{q})=\delta_{n,m}. \qquad (42)$$

Furthermore, it will be assumed that the eigenfunctions are complete. Because of the weighted orthogonality condition, this means that

$$\sum_n \phi^n_{\mathbf{p}}(\mathbf{q},\omega)\,\phi^n_{-\mathbf{p}'}(-\mathbf{q},\omega)=\Delta G_{\mathbf{p}}(\mathbf{q},\omega)\,\delta_{\mathbf{p},\mathbf{p}'}. \qquad (43)$$

The correlation function Φ is given by

$$\Phi_{\mathbf{p},\mathbf{p}'}=\sum_n\frac{\phi^n_{\mathbf{p}}(\mathbf{q},\omega)\,\phi^n_{-\mathbf{p}'}(-\mathbf{q},\omega)}{\lambda^n(\mathbf{q},\omega)}, \qquad (44)$$

and for small ω and $\mathbf{q}$ this sum will be dominated by the first term with nearly vanishing eigenvalue.

Because of the Ward identity,

$$\phi_{\mathbf{p}}^{0}(\mathbf{0},0)=\Delta G_{\mathbf{p}}^{0}/N=[g(\mathbf{p},\Omega+i\epsilon)-g^{*}(\mathbf{p},\Omega+i\epsilon)]/(2iN) \tag{45}$$

is a right eigenfunction of $H^0(\mathbf{0},0)$ with eigenvalue 0. The normalization of ϕ^0 is given by $N^2=\int\Delta G^0 d\mathbf{p}$. For slightly rough interfaces, g can be approximated as a sum of poles corresponding to the normal modes of the layer, plus a continuous density of states which propagate in the upper half-space. The location of the poles is slightly shifted, according to the self-energy M, from where they are in when the interface between the layer and the semi-infinite half-space is flat. The normal modes associated with the poles must decay in the upper half-space, and so the wave vectors of the poles must satisfy $k_m>\Omega/c_1$. In addition to the poles there is a contribution to ΔG^0 from waves which propagate in the upper half-space. In this regime, $p<\Omega/c_1$, g is continuous and given by

$$g(\mathbf{p},\Omega)=\Theta(\mathbf{p},\Omega)/[-i\beta_1(\mathbf{p},\Omega)-Y_0(\mathbf{p},\Omega)-M(\mathbf{p},\Omega)]. \tag{46}$$

Thus the density of states $\Delta G^0=(g-g^*)/(2i)$ becomes

$$\Delta G^0\approx\frac{\Theta(\mathrm{Re}\;\beta_1+\mathrm{Im}\,M)}{[\mathrm{Re}\,\beta_1+\mathrm{Im}\,M]^2+[Y_0+\mathrm{Re}\,M]^2}+\sum_m \pi g_m\delta(p^2-k_m^2). \tag{47}$$

The effect of $\Delta H(0,0)$ is to perturb the eigenvalue associated with $\phi^0=\Delta G^0/N$; the first-order perturbation of the smallest eigenvalue is

$$\lambda_0^0=\frac{\left\langle \phi^0\frac{1}{\Delta G^0}[\mathrm{Re}\,\beta_1]\phi^0\right\rangle}{\int \phi^0\frac{1}{\Delta G^0}\phi^0}=\frac{\int_{p<\Omega/c_1}d\mathbf{p}[\beta_1(\mathbf{p},\Omega)]\Delta G_{\mathbf{p}}^0}{\int d\mathbf{p}\Delta G_{\mathbf{p}}^0}. \tag{48}$$

The eigenvalue λ^0 is positive so that if $1+a>0$, the zero of $\lambda^0(\omega)$ is in the lower-half complex ω plane as expected. For small roughness, the eigenvalue is approximated by

$$\lambda_0^0\approx\int d\mathbf{p}\frac{(\mathrm{Re}\,\beta_1)^2}{Y_0^2+\beta_1^2}\Bigg/\left[\int d\mathbf{p}\frac{\beta_1\Theta(\mathbf{p})}{Y_0^2+\beta_1^2}+\sum_m \pi^2 g_m\right]. \tag{49}$$

In any event we suppose that some approximation can be found for λ^0 and ϕ^0.

Treat $H^1(\mathbf{q},\omega)$ as a small perturbation of H^0 parametrized by the difference wave vector $\mathbf{q}$ and the difference frequency ω, and which vanishes as $\mathbf{q}\to 0$ and $\omega\to 0$. Following the standard Rayleigh perturbation scheme [16], the perturbation of the eigenvalue resulting from nonzero $\mathbf{q}$ and ω is found from

$$\lambda^0(\mathbf{q},\omega)-\lambda^0(\mathbf{0},0)\approx\frac{\int d\mathbf{p}d\mathbf{p}'\,\phi_{\mathbf{p}}^0(0,0)\frac{1}{\Delta G^0}H_{\mathbf{p},\mathbf{p}'}^1\phi_{\mathbf{p}'}^0(\mathbf{q},\omega)}{\int d\mathbf{p}\phi^0\frac{1}{\Delta G^0}\phi_{\mathbf{p}'}^0(\mathbf{q},\omega)}. \tag{50}$$

Of course, the perturbed eigenfunction $\phi_{\mathbf{p}}^0(\mathbf{q},\omega)$ is required if this formula is to be of any use. This situation differs slightly from the standard Rayleigh result; when the operator H_0 is perturbed because $\mathbf{q}$ and ω are nonzero, the weight ΔG^0 is also perturbed to $\Delta G(\mathbf{q},\omega)$. Nevertheless the perturbation procedure described, for example, in Ref. [16] can be used to give the first-order perturbation of the eigenfunction $\delta\phi^0$ as

$$\delta\phi_{\mathbf{p}}^0(\mathbf{q},\omega)=-\int d\mathbf{p}'d\mathbf{p}''\Phi_{\mathbf{p},\mathbf{p}'}^{reg}\frac{1}{\Delta G_{\mathbf{p}'}^0}H_{\mathbf{p}',\mathbf{p}''}^1(\mathbf{q},\omega)\phi_{\mathbf{p}''}^0. \tag{51}$$

The regular contribution to the correlation function, Φ^{reg}, is

$$\Phi_{\mathbf{p},\mathbf{p}'}^{reg}=\sum_{n\neq 0}\frac{\phi_{\mathbf{p}}^n(0,0)\phi_{-\mathbf{p}'}^n(0,0)}{\lambda^n(0,0)-\lambda^0(0,0)}. \tag{52}$$

To determine the coefficient of ω in the expansion of λ^0, again follow Ref. [13] in first setting $\mathbf{q}=0$. Then there are two terms in the perturbation of the operator, one from $-i\Lambda(\mathbf{p},\Omega)\Omega\omega/c(\mathbf{p},\Omega)$ and another from the perturbation of the vertex function $-\omega\partial_{\omega'}U(\mathbf{q},\omega')|_{\mathbf{q}=0,\omega'=0}$. These terms induce a first-order perturbation of the eigenvalue,

$$\lambda^0(0,\omega)=\lambda^0-i\omega\Omega(1+a)\Lambda_0/c_0^2, \tag{53}$$

where

$$\frac{\Lambda_0}{c_0^2}\equiv\left\langle \phi^0\frac{\Lambda(\mathbf{p},\Omega)}{\Delta G^0c^2(\mathbf{p},\Omega)}\phi^0\right\rangle\Bigg/\left\langle \phi^0\frac{1}{\Delta G^0}\phi^0\right\rangle. \tag{54}$$

This is a positive quantity. The characteristic length Λ_0 is defined as

$$N\Lambda_0=\lim_{q\to 0}\int d\mathbf{p}\frac{\mathbf{q}\cdot\mathbf{p}\Lambda(\mathbf{p},\Omega)\phi^0(\mathbf{q})}{-iq^2N} \tag{55}$$

in Eq. (A31). It follows from Eqs. (31) and (50) that the constant a in Eq. (53) is now given by

$$a=-i(c_0^2/(\Lambda_0\Omega))\left\langle \phi^0\frac{1}{\Delta G^0}\partial_{\omega'}U(\mathbf{q},\omega')\bigg|_{\mathbf{q}=0,\omega'=0}\phi^0\right\rangle. \tag{56}$$

Angular brackets indicate integration over wave vectors, for example,

$$\langle\phi^0\rangle\equiv\int d\mathbf{p}\phi_{\mathbf{p}}^0. \tag{57}$$

In the case of dielectric scattering by small particles, Barabanenkov and Ozrin were able to express a in a form which does not involve the derivative of the vertex function U. Van Tiggelen *et al.* went further, showing that a could be expressed in terms of the potential energy within the scattering particles. However, Livdan and Lisyansky left a in terms of $\partial_\omega U$, apparently taking exception to the use of the Ward identity.

To see how the Ward identity is used, note that if $\phi_0/\Delta G^0 = (1/N)(1+\gamma)$, it follows from Eq. (25) and the eigenfunction expansion of Φ that

$$a = -\frac{c_0^2}{2\Lambda_0 \Omega N}\lim_{\omega\to 0}\int d\mathbf{p}\,\frac{S_{\mathbf{p}}(0,\omega)\lambda^0(\omega)}{\omega}\,\frac{1}{\Delta G^0_{\mathbf{p}}}\phi^0 + O(\omega\gamma). \tag{58}$$

The Ward identity [Eq. (23)] then gives

$$a = \lim_{\omega\to 0}\lim_{\mathbf{q}\to 0}\frac{-\lambda_0(\omega)c_0^2}{2\omega\Omega\Lambda_0 N^2}\int d\mathbf{p}d\mathbf{p}'d\mathbf{k}d\mathbf{p}''\langle[y^*_{\mathbf{p}-\mathbf{k}/2,\mathbf{p}'-\mathbf{k}/2}(\Omega - \omega/2) - y_{\mathbf{p}'+\mathbf{k}/2,\mathbf{p}+\mathbf{k}/2}(\Omega+\omega/2)]g_{\mathbf{p}+\mathbf{k}/2,\mathbf{p}''+\mathbf{q}/2} \times(\Omega+\omega/2)g^*_{\mathbf{p}'-\mathbf{k}/2,\mathbf{p}''-\mathbf{q}/2}(\Omega-\omega/2)\rangle \tag{59}$$

If y depends only on wave number differences and depends on frequency only through a quadratic factor, as in dielectric scattering, then Eq. (24) can be used along with Eq. (26) to give

$$a = \frac{c_0^2}{\Lambda_0\Omega^2}\lim_{\omega\to 0}\int d\mathbf{p}[g_0^{-1}\Delta G^0_{\mathbf{p}} - \langle \mathrm{Re}\, g_{\mathbf{p}}\rangle\lambda^0(\omega)] \Big/ \int d\mathbf{p}\Delta G^0_{\mathbf{p}}. \tag{60}$$

This result corresponds to Eq. (12) in Ref. [14] for the case of scattering by dielectric particles when $v(r) = (\Omega^2/c^2)\epsilon(r)$, and where the eigenvalue λ^0 vanishes for small ω. van Tiggelen *et al.* started from this result to derive an identity expressing a in terms of the averaged dielectric constant within the particles. For sound scattering from particles or from a layer which differs in both sound speed and density from its surroundings, Eq. (26) cannot be used. However, the results of Appendix A show that the renormalization of the diffusion coefficient a can be written as

$$a = -1 + \frac{c_0^2}{\Lambda_0 N c_1^2}\int_{\mathbf{p}>\omega/c_1} d\mathbf{p}\int_0^\infty dz e^{-2\,\mathrm{Im}\,\beta_1(\mathbf{p})z}\phi^0_{\mathbf{p}} + \frac{c_0^2}{\Lambda_0 N}\int d\mathbf{p}d\mathbf{k}\int_{-H}^{-h} dz\,\frac{\rho e(z,\mathbf{k})^2}{\rho(z)c(z)^2}|\gamma(\mathbf{k})|^2\langle |T_{1,2}|^2\rangle^{-1}_{\mathbf{k},\mathbf{p}}\phi^0_{\mathbf{p}}$$
$$+ \lim_{\omega\to 0}\frac{(2\pi)^2 c_0^2}{\Lambda_0 N^2}\int d\mathbf{p}\int_{-h}^0 dz\left\langle \psi(z,0|\Omega+\omega/2,\mathbf{p})\,\frac{\rho_0}{\rho(z)c(z)^2}\,\psi^*(z,0|\Omega-\omega/2,\mathbf{p})\right\rangle\lambda^0(\omega). \tag{61}$$

The last integral expresses a as an average of $1/(\rho c^2)$ in the immediate vicinity (between $z=-h$ and $z=0$) of the scattering surface; $-h$ is just below the lowest point of the ensemble of scattering surfaces considered in the averages. In fact, the integral from $-H$ to $-h$ could be removed, and the last integral extended to cover the entire scattering layer, from $z=-H$ to $z=0$. In the integral from $-H$ to $-h$ the fields below the lowest point of the scattering surface have been expanded in modes $e(z,\mathbf{k})$ which satisfy the depth separated Helmholtz equation. Such an expansion allows one to see explicitly that $\langle\psi^*\psi\rangle$ diverges as $1/\lambda(\omega)$ as $\omega\to 0$. However, such an expansion is only possible for $z<-h$.

In this way, a is an average of $\rho_1 c_0^2/(\rho(z)c^2(z))$ both over the scattering layer where the average is weighted by the interior modes, and outside the layer, where the average is weighted by the decaying portion of these modes. If the exterior decaying modes have amplitude 1, then the corresponding interior modes have amplitudes $1/T_{1,2}$, where $T_{1,2}$ is the amplitude for transmission amplitude from inside the layer (medium 2) to the half-space outside the layer (medium 1). This is the source of the factors $1/T_{1,2}$ in Eq. (61). Further discussion of this result is given in Appendix A. In any event, this is how the observation made by van Tiggelen *et al.*, that the renormalization of the diffusion a is the mode weighted average of the dielectric function, is expressed for layered geometry with both sound speed and density contrast across a rough interface.

Now consider the perturbation of λ^0 when $\omega=0$ and $\mathbf{q}$ is small. To first order in $\mathbf{q}$ the perturbation H^1 of H_0 again consists of two terms:

$$H^1_{\mathbf{p},\mathbf{p}'} = i\Lambda(\mathbf{p},\Omega)\mathbf{q}\cdot\mathbf{p}\delta_{\mathbf{p},\mathbf{p}'} - \mathbf{q}\cdot\left.\frac{\partial U_{\mathbf{p},\mathbf{p}'}(\mathbf{q}',0)}{\partial\mathbf{q}'}\right|_{\mathbf{q}'=0}. \tag{62}$$

For reasons of isotropy, first-order contributions to the perturbation of λ^0 vanish, and second-order (in $\mathbf{q}$) terms need to be considered. If ϕ^0 is isotropic, then

$$\Delta\lambda(\mathbf{q})\equiv\lambda^0(\mathbf{q},0)-\lambda^0(0,0)\approx Aq^2$$
$$=\int d\mathbf{p}d\mathbf{p}'\,\phi^0_{\mathbf{p}}\frac{1}{\Delta G^0_{\mathbf{p}}}H^1_{\mathbf{p},\mathbf{p}'}(\mathbf{q},0)\,\delta\phi^0_{\mathbf{p}'}$$
$$+\left\langle\phi^0\frac{1}{\Delta G^0}H^2\phi^0\right\rangle$$
$$=-\left\langle\phi^0\frac{1}{\Delta G^0}H^1\Phi^{reg}\frac{1}{\Delta G^0}H^1\phi^0\right\rangle$$
$$+\left\langle\phi^0\frac{1}{\Delta G^0}H^2\phi^0\right\rangle. \tag{63}$$

In the case of dielectric scattering, when the dielectric function is independent of frequency, the generalized Ward identity holds, and the contribution of the second derivative of U, contained in H^2, vanishes.

H^1 is a sum of two terms, $H^1=i\Lambda(\mathbf{p},\Omega)\mathbf{q}\cdot\mathbf{p}\delta_{\mathbf{p},\mathbf{p}'}-\mathbf{q}(\partial U/\partial\mathbf{q})$, but the four terms that result from $H^1\Phi^{reg}(1/\Delta G^0)H^1$ in Eq. (63) can be rearranged using identity (B13) to give

$$\Delta\lambda(\mathbf{q})\approx\int d\mathbf{p}d\mathbf{p}'\,\phi^0_{\mathbf{p}}\frac{1}{\Delta G^0_{\mathbf{p}}}\Lambda(\mathbf{p},\Omega)\mathbf{q}\cdot\mathbf{p}\Phi^{reg}_{\mathbf{p},\mathbf{p}'}\frac{1}{\Delta G^0_{\mathbf{p}'}}$$
$$\times\Lambda(\mathbf{p}',\Omega)\mathbf{p}'\cdot\mathbf{q}\phi^0_{\mathbf{p}'}-\int d\mathbf{p}\Lambda(\mathbf{p},\Omega)$$
$$\times(\mathbf{q}\cdot\mathbf{p})^2\frac{\partial\,\mathrm{Re}\langle g\rangle}{\partial p^2}$$
$$+2\left\langle\phi^0\frac{1}{\Delta G^0}\mathbf{q}\cdot\frac{\partial U}{\partial\mathbf{q}}\Phi^{reg}\frac{1}{\Delta G^0}i\mathbf{q}\cdot\mathbf{p}\Lambda\,\phi^0\right\rangle$$
$$-\left\langle\phi^0\frac{1}{\Delta G^0}\mathbf{q}\cdot\frac{\partial U}{\partial\mathbf{q}}\Phi^{reg}\frac{1}{\Delta G^0}\mathbf{q}\cdot\frac{\partial U}{\partial\mathbf{q}}\phi^0\right\rangle$$
$$+\left\langle\phi^0\frac{1}{\Delta G^0}H^2\phi^0\right\rangle. \tag{64}$$

In the case of no density contrast, which is formally the same as the case of dielectric scattering without dispersion considered by Barabanenkov and Ozrin, the generalized Ward identity can be invoked to show that the third and fourth terms vanish, and that all that remains of the last term is the part resulting from $g_0^{-1}-g_0^{*-1}$, i.e., the part that is peculiar to the layered geometry considered here.

The first two terms can be compared to Eq. (22) in Ref. [13]. This result, however, in no way depends on the generalized Ward identity. Kogan and Kaveh indicated that the correction to the diffusion resulting from a is a result of the time delay associated with resonant wave scattering. Similar reasoning shows that the correction to the diffusion resulting from the term in $\partial U/\partial\mathbf{q}$ can be associated with large scatterers, so that energy enters the scatterer at one location and emerges at another. In Voronovich's [15] treatment of scattering in a layer, this nonlocal character of the scattering shows itself in the appearance of the skip distance.

An alternate way of writing the perturbation of λ for nonzero $\mathbf{q}$ without invoking identity (B13) is

$$\Delta\lambda(\mathbf{q})\approx\left\langle\phi^0\frac{1}{\Delta G^0}i\mathbf{q}\cdot\mathbf{p}\Lambda\,\delta\phi(\mathbf{q})\right\rangle$$
$$+\frac{iq_iq_j}{4}\left\langle\phi^0\frac{1}{\Delta G^0}\frac{\partial^2g_0^{-1}-g_0^{*-1}}{\partial q_i\partial q_j}\phi^0\right\rangle$$
$$-\left\langle\phi^0\frac{1}{\Delta G^0}\mathbf{q}\cdot\frac{\partial U}{\partial q}\delta\phi(\mathbf{q})\right\rangle$$
$$-\left\langle\phi^0\frac{1}{\Delta G^0}\frac{\partial^2U}{\partial q_i\partial q_j}(q_iq_j/2)\phi^0\right\rangle. \tag{65}$$

The first term on the right gives the standard result when the generalized Ward identity holds. It contains a term in the derivative of U which was described explicitly by Livdan and Lisyansky [17], and which became the expression involving $\partial\,\mathrm{Re}\,g/\partial p^2$ in Barabanenkov and Ozrin. From the definition of Λ_0 given in Eq. (A31), as $q\to0$ the first term in Eq. (65) becomes

$$\left\langle\phi^0\frac{1}{\Delta G^0}i\mathbf{q}\cdot\mathbf{p}\Lambda\,\delta\phi(\mathbf{q})\right\rangle\to\Lambda_0q^2. \tag{66}$$

The second term on the right of Eq. (65), which involves second derivatives of g_0^{-1}, is problematic; it appears to be divergent. If only pole contributions to ΔG^0 are retained, as in the work of Sánchez-Gil *et al.* [4], this term never appears. The divergence arises from wave vectors near grazing, and one can only suppose that these need special treatment. Investigation of this term will not be attempted here.

The last terms in Eq. (65) can be expressed in terms of ''kinetic-energy-like'' integrals over the layer using the results of Appendix A,

$$-\left\langle\phi^0\frac{1}{\Delta G^0}\mathbf{q}\cdot\frac{\partial U}{\partial q}\delta\phi(\mathbf{q})\right\rangle$$
$$-\left\langle\phi^0\frac{1}{\Delta G^0}\frac{\partial^2U}{\partial q_i\partial q_j}(q_iq_j/2)\phi^0\right\rangle\to A_1q^2, \tag{67}$$

where A_1 is given in Eq. (A54).

Finally, neglecting the problematic terms in $\partial^2g_0^{-1}$,

$$A=\Lambda_0+A_1$$
$$=1/N\int_{Q>\Omega/c_1}d\mathbf{Q}\int_0^\infty dz(Q^2/2)e^{-2\,\mathrm{Im}\,\beta_1(\mathbf{Q})z}B_Q\phi^0_{\mathbf{Q}}$$
$$+1/N\int d\mathbf{Q}\int_{-H}^{-h}dz(Q^2/2)\frac{\rho_1}{\rho(z)}B_Q^{int}(z)\phi^0_{\mathbf{Q}}$$
$$+\frac{-\Lambda_0}{2iN}\lim_{q\to0}\int d\mathbf{p}(2\pi)^2\rho_1\int_{-h}^0dz[\mathbf{F}(z|\Omega,0,\mathbf{p},\mathbf{q}/2)$$
$$+\mathbf{F}^*(z|\Omega,0,\mathbf{p},-\mathbf{q}/2)]\cdot\mathbf{q}\frac{\lambda(\mathbf{q},0)}{q^2\Delta G_{\mathbf{p}}(\mathbf{q})}\phi^0_{\mathbf{p}}(\mathbf{q}). \tag{68}$$

The integration over $-h$ to 0 could be dropped, and the last integral could be extended over the entire layer. Since **F** is the energy flux when $q \to 0$, it seems that A and the diffusion are intimately related to the flux density in the layer, but the limiting procedures called for make the connection a bit obscure.

In contrast to the case of scattering by small particles, the corrections a and A_1 are not small (proportional to the number density of scatterers). Rather the ''1'' in $1+a$ is completely canceled, as is the Λ_0 in $A=\Lambda_0+A_1$.

IV. SUMMARY

Equations (37), (61), and (68) can be combined to give an admittedly unwieldy expression for the diffusion constant D. Likewise, the decay time τ follows from Eqs. (38), (61), and (68).

The main point of this paper is that connection between the renormalization of the diffusion coefficient and the energylike or fluxlike integrals does not depend on the generalized Ward identity of Barabanenkov and Ozrin, which holds only for local dielectric scattering. Furthermore, the second renormalization of the diffusion, considered by Livdan and Lisyansky is also related to energy flux integrals. This reseach is motivated by the observation that simply stated relationships, such as that discovered by van Tiggelen *et al.*, probably have some wide generality, and that the arguments of Kogan and Kaveh seem to capture the essence of wave diffusion.

APPENDIX A

The purpose of this appendix is to demonstrate the connection between the quantity S and energylike averages over both the interior and exterior of the scattering layer.

Begin by considering a generalized flux within the layer. Define **W** by

$$\mathbf{W}(\mathbf{r},\mathbf{q}|\Omega,\omega,\mathbf{K},\mathbf{k}) \equiv e^{-i\mathbf{q}\cdot\mathbf{R}}\Bigg[\psi^*(\mathbf{r}|\Omega-\omega/2,\mathbf{K}-\mathbf{k}/2) \times\frac{1}{\rho(\mathbf{r})}\nabla\psi(\mathbf{r}|\Omega+\omega/2,\mathbf{K}+\mathbf{k}/2) - \frac{1}{\rho(\mathbf{r})}\nabla\psi^*(\mathbf{r}|\Omega-\omega/2,\mathbf{K}-\mathbf{k}/2) \times\psi(\mathbf{r}|\Omega+\omega/2,\mathbf{K}+\mathbf{k}/2)\Bigg]. \tag{A1}$$

If ψ is a pressure field, then the corresponding velocity field is $\mathbf{v}=-i\nabla\psi/(\rho\omega)$. Thus if $\mathbf{k}=0$ and $\omega=0$, the term in square brackets is proportional to the energy flux.

These fields ψ depend implicitly on the surface roughness $h(\mathbf{r})$, the sound speed within the layer, $c(\mathbf{r})$, and the density within the layer, $\rho(\mathbf{r})$. If each of these quantities is translated horizontally by a vector **a** the fields are likewise translated and multiplied by a phase factor, so that

$$\psi(\mathbf{r}|\mathbf{K},[h(\mathbf{r}-\mathbf{a}),\rho(\mathbf{r}-\mathbf{a}),c(\mathbf{r}-\mathbf{a})]) = e^{i\mathbf{K}\cdot\mathbf{a}}\psi(\mathbf{r}-\mathbf{a}|\mathbf{K},[h(\mathbf{r}),\rho(\mathbf{r}),c(\mathbf{r})]). \tag{A2}$$

If there is statistical homogeneity in the horizontal directions, it follows that

$$\langle\psi^*(\mathbf{r}|\mathbf{K}-\mathbf{k}/2)\mathbf{v}(\mathbf{r}|\mathbf{K}+\mathbf{k}/2)\rangle = e^{i\mathbf{k}\cdot\mathbf{a}}\langle\psi^*(\mathbf{r}-\mathbf{a}|\mathbf{K}-\mathbf{k}/2)\mathbf{v}(\mathbf{r}-\mathbf{a}|\mathbf{K}+\mathbf{k}/2)\rangle. \tag{A3}$$

This must hold for all horizontal translations **a**. In particular, if $\mathbf{a}=\mathbf{R}$, the horizontal projection of **r**, then

$$\begin{aligned}\langle\psi^*(\mathbf{r}|\Omega-\omega/2,\mathbf{K}-\mathbf{k}/2)\mathbf{v}(\mathbf{r}|\Omega+\omega/2,\mathbf{K}+\mathbf{k}/2)\rangle &= e^{i\mathbf{k}\cdot\mathbf{R}}\langle\psi^*(z,\mathbf{R}=0|\Omega-\omega/2,\mathbf{K}-\mathbf{k}/2) \times\mathbf{v}(z,\mathbf{R}=0|\Omega+\omega/2,\mathbf{K}+\mathbf{k}/2)\rangle \\ &\equiv e^{i\mathbf{k}\cdot\mathbf{R}}\mathbf{F}(z|\mathbf{K},\mathbf{k},\Omega,\omega).\end{aligned} \tag{A4}$$

The real part of $\mathbf{F}(z|\mathbf{K}v,0,\Omega,0)$ is the average energy flux density.

The strategy for connecting interior fields to surface fields is to integrate the divergence of **W** over the volume of the layer which extends from $z=-H$ to $z=0$, which is just above the rough interface between the layer and the half-space above. Then apply Green's theorem. Assume boundary conditions on the lower surface that insure that $\mathbf{n}\cdot\mathbf{W}=0$ there. The average of the divergence of **W** is given by

$$\langle\nabla\cdot\mathbf{W}\rangle = e^{-i(\mathbf{q}-\mathbf{k})\cdot\mathbf{R}}\Bigg\{-i\mathbf{q}\cdot[i(\Omega+\omega/2)\mathbf{F}(z|\mathbf{K},\mathbf{k},\Omega,\omega) + i(\Omega-\omega/2)\mathbf{F}^*(z|\mathbf{K},-\mathbf{k},\Omega,-\omega)] - \left\langle\psi^*(z,0)\frac{2\Omega\omega}{\rho(z,0)c^2(z,0)}\psi(z,0)\right\rangle\Bigg\}. \tag{A5}$$

The angular brackets indicate averages over all realizations of h, c, and ρ. Integrating over $d\mathbf{R}$ gives $(2\pi)^2\delta(\mathbf{q}-\mathbf{k})$. Hence a subsequent integration over **q** gives

$$\begin{aligned}\int d\mathbf{R}d\mathbf{q}\langle\mathbf{W}(0,\mathbf{R})\rangle\cdot\hat{\mathbf{z}} &= (2\pi)^2\int_{-H}^{0}dz\,\mathbf{k}\cdot[(\Omega+\omega/2)\mathbf{F}(z|\Omega,\omega\mathbf{K},\mathbf{k}/2) + (\Omega-\omega/2)\mathbf{F}^*(z|\Omega,-\omega,\mathbf{K},-\mathbf{k}/2)] \\ &\quad -(2\pi)^2\int_{-H}^{0}dz\left\langle\psi^*(z,0)\frac{2\Omega\omega}{\rho(z,0)c^2(z,0)}\psi(z,0)\right\rangle.\end{aligned} \tag{A6}$$

Now evaluate the surface integral directly using the definition of **W** and expressing the normal derivatives in terms of the admittance [Eq. (4)]:

$$\int d\mathbf{R}\langle \mathbf{W}(0,\mathbf{R})\rangle\cdot\hat{\mathbf{z}} = \int d\mathbf{R}e^{-i\mathbf{q}\cdot\mathbf{R}}\left\langle \psi^*\frac{1}{\rho}\partial_z\psi - \frac{1}{\rho}\partial_z\psi^*\psi\right\rangle = \int d\mathbf{R}e^{-i\mathbf{q}\cdot\mathbf{R}}\left\langle -\psi^*\frac{1}{\rho}Y\psi + \frac{1}{\rho}Y^*\psi^*\psi\right\rangle = \int d\mathbf{Q}d\mathbf{P}\frac{1}{\rho}\langle[Y^*(\mathbf{P}-\mathbf{q}/2,\mathbf{Q}-\mathbf{q}/2) - Y(\mathbf{Q}+\mathbf{q}/2,\mathbf{P}+\mathbf{q}/2)]\hat{\psi}^*(0,\mathbf{Q}-\mathbf{q}/2|\mathbf{K}-\mathbf{k}/2) \times\hat{\psi}(0,\mathbf{P}+\mathbf{q}/2|\mathbf{K}+\mathbf{k}/2)\rangle. \tag{A7}$$

The horizontal Fourier transform of the field in this expression is

$$\hat{\psi}(0,\mathbf{Q}|\mathbf{K}+\mathbf{k}/2) = \int\frac{d\mathbf{R}}{2\pi}e^{-i\mathbf{Q}\cdot\mathbf{R}}\psi(0,\mathbf{R}|\mathbf{K}+\mathbf{k}/2,\Omega+\omega/2). \tag{A8}$$

Suppose that the field on the surface $z=0$ is the sum of an incident plane wave and the corresponding reflected plane waves normalized by the vertical wave number, i.e.,

$$\hat{\psi}(0,\mathbf{Q}|\mathbf{K}) = \frac{\delta(\mathbf{Q}-\mathbf{K})+R_{\mathbf{Q},\mathbf{K}}}{-2i\beta_1(\mathbf{K})} = g(\mathbf{Q},\mathbf{K}). \tag{A9}$$

Then the surface integral of $\mathbf{W}$ involves g^*g . Noting that the admittance $Y=Y_0+y$ is the total admittance and that S is expressed, through the Ward identity, in terms of the fluctuation of the admittance y we can write an alternative result for the integral in Eq. (A7):

$$\int d\mathbf{R}d\mathbf{q}\langle\mathbf{W}(0,\mathbf{R})\rangle\cdot\hat{\mathbf{z}} = \frac{1}{\rho}S_{\mathbf{K}}(\mathbf{k}|\Omega,\omega) + \frac{1}{\rho}\int d\mathbf{P}[Y_0^*(\Omega-\omega/2,\mathbf{P}-\mathbf{k}/2) - Y_0(\Omega+\omega/2,\mathbf{P}+\mathbf{k}/2)]\Phi_{\mathbf{P},\mathbf{K}}(\mathbf{k}|\Omega,\omega). \tag{A10}$$

Equations (A6) and (A10) together give

$$S_{\mathbf{K}}(\mathbf{k}|\Omega,\omega) = \int d\mathbf{P}[Y_0(\Omega+\omega/2,\mathbf{P}+\mathbf{k}/2) - Y_0^*(\Omega-\omega/2,\mathbf{P}-\mathbf{k}/2)] \times\Phi_{\mathbf{P},\mathbf{K}}(\mathbf{k}|\Omega,\omega) + (2\pi)^2\rho\int_{-H}^{0}dz\mathbf{k}\cdot[(\Omega+\omega/2) \times\mathbf{F}(z|\Omega,\omega,\mathbf{K},\mathbf{k}/2) + (\Omega-\omega/2)\mathbf{F}^*(z|\Omega,-\omega,\mathbf{K}, -\mathbf{k}/2)] - (2\pi)^2\rho\int_{-H}^{0}dz\left\langle \psi^*(z,0|\Omega-\omega/2,\mathbf{K} -\mathbf{k}/2)\frac{2\Omega\omega}{\rho(z,0)c^2(z,0)}\psi(z,0|\Omega+\omega/2,\mathbf{K}+\mathbf{k}/2)\right\rangle. \tag{A11}$$

1. Contribution of $1/\rho c^2$

Equation (A11) suffices to relate the potential energy $(1/(\rho c^2))$ and the ''kinetic energy'' $p^2/2$ inside the layer to the surface quantity S. In this subsection we deal with the renormalization of the diffusion coefficient a which comes from the limit $\omega\to 0$ after $\mathbf{q}$ has been set to zero. In Eq. (62), a is expressed in terms of

$$\lim_{\omega\to 0}\int d\mathbf{p}\,\frac{S_{\mathbf{p}}(0|\Omega,\omega)}{\omega}\frac{\lambda^0(\omega)}{\Delta G_{\mathbf{p}}^0}\phi_{\mathbf{p}}^0 = \int d\mathbf{p}\,\frac{\partial Y_0(\Omega)}{\partial\Omega}\phi_{\mathbf{p}}^0 - (2\pi)^2\lim_{\omega\to 0}\int\int_{-H}^{0}dz\left\langle \psi^*(z,0|\Omega -\omega/2)\frac{2\Omega\rho}{\rho(z,0)c^2(z,0)}\psi(z,0|\Omega+\omega/2)\right\rangle\frac{\lambda^0(\omega)}{\Delta G_{\mathbf{p}}^0}\phi_{\mathbf{p}}^0 d\mathbf{p}. \tag{A12}$$

In the last expression, if the eigenvalue vanishes as $\omega\to 0$, then the average of $\psi^*\psi$ will diverge as $1/\lambda^0$, since the long-time behavior of the fields within the layer must track the long-time behavior of the fields just outside the layer. As a result, the limit can be nonvanishing even though λ vanishes.

Use

$$\frac{\partial Y_0(\mathbf{p},\Omega)}{\partial\Omega} = +2\frac{\Lambda(\mathbf{p},\Omega)\Omega}{c(\mathbf{p},\Omega)^2} + \frac{\partial\,\mathrm{Im}\,\beta_1}{\partial\Omega}, \tag{A13}$$

which follows from the definition of $\Lambda(\mathbf{p},\Omega)/c(\mathbf{p},\Omega)^2$ in terms of the derivatives of $g_0^{-1} = -i\beta_1 - Y_0$, to write the first integral as

$$\int d\mathbf{p}\frac{\partial Y_0(\Omega)}{\partial\Omega}\phi_{\mathbf{p}}^0 = \frac{2\Omega\Lambda_0 N}{c_0^2}\left(1+\frac{c_0^2}{\Lambda_0 N}\int d\mathbf{p}\frac{\partial\,\mathrm{Im}\,\beta_1}{\partial(\Omega^2)}\phi_{\mathbf{p}}^0\right). \tag{A14}$$

Fields that decay away from the surface do so as $\exp(-\mathrm{Im}\,\beta_1 z)$, and the last integral can be written as the z integral of the decaying fields using

$$\frac{\partial\,\mathrm{Im}\,\beta_1}{\partial(\Omega^2)} = -\frac{1}{2c_1^2\,\mathrm{Im}\,\beta_1} = -\frac{1}{c_1^2}\int_0^\infty dz e^{-2\,\mathrm{Im}\,\beta_1 z}. \tag{A15}$$

Using this result in Eq. (A12), and then Eq. (A12) in Eq. (62), gives

$$a = -1 + \frac{c_0^2}{\Lambda_0 N}\int_{p>\Omega/c_1}d\mathbf{p}\int_0^\infty dz\,\frac{1}{c_1^2}\,e^{-2\,\mathrm{Im}\,\beta_1(\mathbf{p},\Omega)z}\phi_{\mathbf{p}}^0 + \frac{(2\pi)^2c_0^2}{\Lambda_0 N}\int d\mathbf{p}\int_{-H}^{0}dz\left\langle \psi^*(z,0|\Omega-\omega/2,\mathbf{p}) \times\frac{\rho}{\rho(z,0)c^2(z,0)}\psi^*(z,0|\Omega-\omega/2,\mathbf{p})\right\rangle\frac{\lambda^0(\omega)}{\Delta G_{\mathbf{p}}^0}\phi_{\mathbf{p}}^0. \tag{A16}$$

In this way a is a field-weighted average of $1/(\rho c^2)$ within and outside the scattering layer. In the last integral, the limit $\omega\to 0$ is understood, and one needs to assume that since $\psi(z,\mathbf{R}=0|\Omega\pm\omega/2,\mathbf{p})$ is driven by its value on the surface $z=0$, namely, $(1/(2\pi))\int d\mathbf{Q} g(\mathbf{Q},\mathbf{p})$, that the correlation function $\langle\psi^*\psi\rangle$ will diverge as $\omega\to 0$ in the same way that $\Phi=\langle gg^*\rangle$ diverges, i.e., as $1/\lambda^0(\omega)$. It is possible to see how this can happen by expressing the fields beneath the lowest excursions of the scattering surfaces (in the region $z\leqslant -h$) in terms of fields which satisfy the depth-separated wave equation

$$\frac{d}{dz}\frac{1}{\rho(z)}\frac{d}{dz}e(z,\mathbf{Q})=-\frac{1}{\rho(z)}\left(\frac{\Omega^2}{c(z)^2}-\mathbf{Q}^2\right)e(z,\mathbf{Q}), \tag{A17}$$

with boundary conditions

$$e(-H,\mathbf{Q})=0 \tag{A18}$$

and

$$\frac{d}{dz}e(z,\mathbf{Q})|_{z=-H}=1. \tag{A19}$$

Then the fields ψ can be written

$$\psi(z,\mathbf{R}|\Omega,\mathbf{K})=\frac{1}{2\pi}\int d\mathbf{Q}\exp(i\mathbf{Q}\cdot\mathbf{R})e(z,\mathbf{Q})\alpha(\mathbf{Q}|\mathbf{K})/(-2i\beta_1(\mathbf{K})) \tag{A20}$$

for $-H\leqslant z\leqslant -h$. The functions $\alpha(\mathbf{Q}|\mathbf{K})$ are simply the expansion coefficients of ψ in terms of the basis functions e. The factor $1/(-2i\beta_1)$ is made explicit because $g=(1+R)/(-2i\beta_1)$.

In the usual fashion, statistical homogeneity means that we can write

$$\begin{aligned}&\langle\alpha(\mathbf{p}+\mathbf{q}/2|\Omega+\omega/2,\mathbf{p}'+\mathbf{q}'/2)\alpha^*\\&\times(\mathbf{p}-\mathbf{q}/2|\Omega-\omega/2,\mathbf{p}'-\mathbf{q}'/2)\rangle\\&=\Gamma_{\mathbf{p},\mathbf{p}'}(\mathbf{q}|\Omega,\omega)\,\delta(\mathbf{q}-\mathbf{q}').\end{aligned} \tag{A21}$$

If the density and sound speed within the layer are independent of surface statistics, then

$$\begin{aligned}&\lim_{\omega\to 0}\left\langle\psi^*(z,0|\Omega-\omega/2,\mathbf{p})\frac{\rho}{\rho(z,0)c^2(z,0)}\psi^*(z,0|\Omega-\omega/2,\mathbf{p})\right\rangle\lambda^0(\omega)\\&=\lim_{\omega\to 0}\int\frac{d\mathbf{q}}{(2\pi)^2}e(\mathbf{q},z)^2\left\langle\frac{\rho}{\rho(z,0)c^2(z,0)}\right\rangle\Gamma_{\mathbf{q},\mathbf{p}}(0|\Omega,\omega)\lambda^0(\omega)/4\beta_1(\mathbf{p},\Omega+\omega/2)\beta_1^*(\mathbf{p},\Omega-\omega/2).\end{aligned} \tag{A22}$$

The next issue to resolve is how Γ is related to Φ. R is the *net* scattering amplitude from medium 1 back into medium 1. The net field scattered is the result of multiple scattering between the rough interface and the (possibly refracting) layer bounded below at $z=-H$, which will now be described.

Between $z=-h$ and the rough interface assume that the density and sound speed are constant (see Fig. 1). A plane wave which is incident from above on the layer between $z=-h$ and $z=-H$ from a semi-infinite half-space with $c=c_2$ and $\rho=\rho_2$ is reflected with an amplitude $V(\mathbf{Q})$, so that the total field associated with the plane wave is

$$\begin{aligned}\phi_{pw}(z,\mathbf{R}|\mathbf{Q})=&[\exp(-i\beta_2(\mathbf{Q}))z\\&+\exp(i\beta_2(\mathbf{Q}))V(\mathbf{Q})]\exp(i\mathbf{Q}\cdot\mathbf{R}).\end{aligned} \tag{A23}$$

These fields ϕ_{pw} can be used as a basis to write the scattered field in the presence of the rough interface as

$$\begin{aligned}\psi(z,\mathbf{R}|\mathbf{K})=&(1/(2\pi))\int d\mathbf{Q}\exp(-i\mathbf{Q}\cdot\mathbf{R})\\&\times[\exp(-i\beta_2(\mathbf{Q})z)+\exp(i\beta_2(\mathbf{Q})z)V(\mathbf{Q})]\\&\times T(\mathbf{Q}|\mathbf{K})/-2i\beta_1(\mathbf{K}).\end{aligned} \tag{A24}$$

This expansion is only valid between the rough interface and $z=-h$. In fact, one would have to invoke the Rayleigh hypothesis to use this expansion up to the lower side of the rough interface. The expansion coefficients $T(\mathbf{Q}|\mathbf{K})$ will be described below.

The field $\phi(z,\mathbf{R}|\mathbf{Q})$, which is the continuation of the plane wave state ϕ_{pw} into the transversely homogeneous layer below $z=-h$, must be a multiple of $\exp(i\mathbf{Q}\cdot\mathbf{R})e(z,\mathbf{Q})$:

$$\phi(z,\mathbf{R})=\exp(i\mathbf{Q}\cdot\mathbf{R})e(z,\mathbf{Q})\,\gamma(\mathbf{Q}). \tag{A25}$$

Continuity of the field and its normal derivative across the imaginary surface at $z=-h$ determine both the reflection coefficient $V(\mathbf{Q})$ and the ''transmission'' coefficient $\gamma(\mathbf{Q})$. The ratio $(d/dz)e(z,\mathbf{Q})|_{z=-h}/e(-h,\mathbf{Q})=A(\mathbf{Q})$ is the admittance of the surface $z=-h$, and V is given by

$$V(\mathbf{Q})=\exp(i2\beta_2(\mathbf{Q})h)\frac{1+A(\mathbf{Q})/i\beta_2(\mathbf{Q})}{1-A(\mathbf{Q})/i\beta_2(\mathbf{Q})}. \quad \text{(A26)}$$

The ''transmission coefficient'' γ is then given by

$$\gamma(\mathbf{Q})=[\exp(i\beta_2(\mathbf{Q})h)+\exp(-i\beta_2(\mathbf{Q})h)V(\mathbf{Q})]/e(-h|\mathbf{Q}). \quad \text{(A27)}$$

It follows that in Eq. (A20),

$$\alpha(\mathbf{Q})=\gamma(\mathbf{Q})T(\mathbf{Q}|\mathbf{K}). \quad \text{(A28)}$$

Now express the net reflection amplitude R in terms of T. The reflection amplitude $R(\mathbf{Q}|\mathbf{K})$ is the result of the initial encounter with the interface, described by $R_{1,1}(\mathbf{Q}|\mathbf{K})$ plus whatever part of the upgoing waves in selvage region $-h<z<h(\mathbf{R})$ is transmitted by the interface. The amplitude of the upgoing waves is $V(\mathbf{P})T(\mathbf{P}|\mathbf{K})$ from Eq. (A24), and the portion that is transmitted up into medium 1 with wave vector $\mathbf{Q}$ is $\int d\mathbf{P}T_{1,2}(\mathbf{Q}|\mathbf{P})V(\mathbf{P})T(\mathbf{P}|\mathbf{K})$. Hence the net reflection amplitude is given by

$$R(\mathbf{Q}|\mathbf{K})=R_{1,1}(\mathbf{Q}|\mathbf{K})+\int d\mathbf{P}T_{1,2}(\mathbf{Q}|\mathbf{P})V(\mathbf{P})T(\mathbf{P}|\mathbf{K}). \quad \text{(A29)}$$

On the other hand, the downgoing waves in the selvage region [whose amplitudes are $T(\mathbf{Q}|\mathbf{K})$ in Eq. (A24)] are the result of transmission of the plane wave incident from above [described by $T_{2,1}(\mathbf{Q}|\mathbf{K})$] plus the internal reflection of the upgoing waves in the selvage region, described by $\int d\mathbf{P}R_{2,2}(\mathbf{Q}|\mathbf{P})V(\mathbf{P})T(\mathbf{P}|\mathbf{K})$. It follows that T satisfies

$$T(\mathbf{Q}|\mathbf{K})=T_{2,1}(\mathbf{Q}|\mathbf{K})+\int d\mathbf{P}R_{2,2}(\mathbf{Q}|\mathbf{P})V(\mathbf{P})T(\mathbf{P}|\mathbf{K}). \quad \text{(A30)}$$

The formal solution of this equation is

$$T=\frac{1}{1-R_{2,2}V}T_{2,1}. \quad \text{(A31)}$$

Using this result in Eq. (A29) gives

$$1+R=1+R_{1,1}+T_{1,2}V\frac{1}{1-R_{2,2}V}T_{2,1}=1+R_{1,1}+T_{1,2}VT. \quad \text{(A32)}$$

Only the last term involves multiple scattering. In the hydrodynamic limit, we neglect the single-scatter contribution, and write the the surface correlation function (schematically) as

$$\Phi=\langle gg^*\rangle\approx\langle(T_{1,2}VT)(T_{1,2}VT)^*\rangle/(4\beta_1\beta_1^*). \quad \text{(A33)}$$

Following Voronovich [15], we argue that the essence of the ladder approximation, the primary ingredient of the Bethe-Salpeter equation, is the assumption of uncorrelated successive scattering events. The same assumption can be used to factor the average on the right of the previous equation, giving

$$\Phi\approx\langle T_{1,2}T_{1,2}^*\rangle VV^*\langle TT^*\rangle/(4\beta_1\beta_1^*)$$
$$=\langle T_{1,2}T_{1,2}^*\rangle\frac{1}{\gamma\gamma^*}\Gamma\Big/(4\beta_1\beta_1^*). \quad \text{(A34)}$$

The last line follows because it is assumed that $VV^*=1$, which is the case, for example, when $V=-\exp(2i\beta_2H)$. Furthermore $\langle\alpha\alpha^*\rangle=\Gamma=\gamma\gamma^*\langle TT^*\rangle$. Let $\langle|T_{1,2}|^2\rangle^{-1}$ denote the operator inverse of $\langle T_{1,2}T_{1,2}^*\rangle$. Then the correlation function of T can be written

$$\Gamma/(4\beta_1\beta_1^*)=\gamma\gamma^*\langle T_{1,2}T_{1,2}^*\rangle^{-1}\Phi, \quad \text{(A35)}$$

and the correlation of fields within the layer below $z=-h$ can be written for small ω as

$$\left\langle \psi^*(z,0|\Omega-\omega/2,\mathbf{p})\frac{\rho}{\rho(z,0)c^2(z,0)}\psi^*(z,0|\Omega-\omega/2,\mathbf{p})\right\rangle$$
$$=\int\frac{d\mathbf{q}}{(2\pi)^2}e(z,\mathbf{q})^2|\gamma(\mathbf{q})|^2\frac{\rho}{\rho(z,0)c^2(z,0)}$$
$$\times\langle|T_{1,2}|^2\rangle^{-1}_{\mathbf{q},\mathbf{p}'}\Phi_{\mathbf{p}',\mathbf{p}}(0|\Omega,\omega)d\mathbf{p}'. \quad \text{(A36)}$$

Combining Eqs. (A36) and (A16) and the eigenvalue expansion of Φ yields Eq. (61) of the text. The eigenfunction ϕ^0 represents a certain incoherent distribution of wave vectors. The two integrals in Eq. (61) are averages inside and outside the layer of $\rho c_0^2/\rho(z)c(z)^2$.

2. Contribution of p^2

The constant A in Eq. (30) for the eigenvalue $\lambda(\mathbf{q},\omega)$ is determined from

$$A=\lim_{q\to0}\frac{\Delta\lambda(\mathbf{q},0)}{q^2}, \quad \text{(A37)}$$

with $\Delta\lambda(\mathbf{q},0)$ given by Eq. (65) of the text. The aim of this section is to determine the contribution A_1 to A from the last two terms of Eq. (65). Analogously to Eq. (62) for a, this contribution to A can be expressed in terms of S and then, using Eq. (A11), in terms of depth integrals.

From the definition of S [Eq. (27)] and the eigenfunction expansion of Φ [Eq. (48)], it follows that

$$A_1=\frac{-1}{2iN}\lim_{q\to0}\int d\mathbf{p}\frac{S_{\mathbf{p}}(\mathbf{q}|\Omega,0)}{q^2}\frac{\lambda(\mathbf{q},0)}{\Delta G_{\mathbf{p}}^0(\mathbf{q})}\phi_{\mathbf{p}}^0(\mathbf{q}), \quad \text{(A38)}$$

in the same way that Eq. (62) follows from Eq. (60). On the other hand, Eq. (A11) for $\omega=0$ and $\mathbf{q}\to0$ gives

$$S_{\mathbf{p}}(\mathbf{q}|\Omega,0)=\int d\mathbf{p}'\frac{\partial Y_0(\Omega,\mathbf{p}')}{\partial\mathbf{p}'}\Phi_{\mathbf{p}',\mathbf{p}}(\mathbf{q})$$
$$+(2\pi^2)\rho_1\int_{-H}^{0}\Omega\mathbf{q}\cdot[\mathbf{F}(z|\Omega,0,\mathbf{p},\mathbf{q})$$
$$+\mathbf{F}^*(z|\Omega,0,\mathbf{p},-\mathbf{q})]. \quad \text{(A39)}$$

Let A_1^Y be the contribution to A_1 from the first integral in S, i.e., the contribution of Y_0. Following steps analogous to those leading from Eq. (A11) to Eq. (A15), we write

$$\mathbf{q}\cdot\frac{\partial Y_0}{\partial\mathbf{p}}=2\mathbf{q}\cdot\mathbf{p}\frac{\partial Y_0}{\partial p^2}=2\mathbf{q}\cdot\mathbf{p}\left[-\Lambda(\mathbf{p},\Omega)+\frac{\partial\,\mathrm{Im}\,\beta_1}{\partial p^2}\right]. \tag{A40}$$

Then, using the eigenfunction expansion of Φ, the contribution A_1^Y can be written

$$A_1^Y=\frac{-1}{N}\lim_{q\to 0}\int d\mathbf{p}\frac{\mathbf{q}\cdot\mathbf{p}}{iq^2}\left[-\Lambda(\mathbf{p},\Omega)+\frac{\partial\,\mathrm{Im}\,\beta_1}{\partial p^2}\right]\phi_{\mathbf{p}}^0(\mathbf{q}). \tag{A41}$$

The first integral is a wave-vector average of Λ, and motivates the definition

$$N\Lambda_0=\lim_{\mathbf{q}\to 0}\int d\mathbf{p}\frac{\mathbf{q}\cdot\mathbf{p}\Lambda(\mathbf{p},\Omega)}{-iq^2}\phi_{\mathbf{p}}^0(\mathbf{q}). \tag{A42}$$

Using this definition and

$$\frac{\partial\,\mathrm{Im}\,\beta_1}{\partial p^2}=\frac{1}{2\,\mathrm{Im}\,\beta_1}=\int_0^\infty dz e^{-2\,\mathrm{Im}\,\beta_1 z}, \tag{A43}$$

A_1^Y becomes

$$A_1^Y=-\Lambda_0-\lim_{q\to 0}\frac{1}{N}\int d\mathbf{p}\int_0^\infty dz e^{-2\,\mathrm{Im}\,\beta_1 z}\frac{\mathbf{q}\cdot\mathbf{p}}{iq^2}\delta\phi_{\mathbf{p}}^0(\mathbf{q}). \tag{A44}$$

Following Livdan and Lisyansky [8] and others, we write the perturbation of the eigenfunction in the form

$$\delta\phi_{\mathbf{p}}^0(\mathbf{q})=-i\mathbf{q}\cdot\mathbf{p}B_p\phi_{\mathbf{p}}^0(0), \tag{A45}$$

so that

$$A_1^Y=-\Lambda_0+\frac{1}{N}\int d\mathbf{p}\int_0^\infty dz(p^2/2)e^{-2\,\mathrm{Im}\,\beta_1 z}B_p\phi_p^0 \tag{A46}$$

and

$$\Lambda_0=\int d\mathbf{p}(p^2/2)\Lambda(\mathbf{p},\Omega)B_p\phi_{\mathbf{p}}^0\bigg/\int d\mathbf{p}\phi_{\mathbf{p}}^0. \tag{A47}$$

Recall that $\Lambda(\mathbf{p},\Omega)>0$, so that if B_p is positive, as will be assumed, then Λ_0 is positive.

Now treat the contribution to A_1 that comes from the integral of $\mathbf{F}$ in S. Follow the steps that yield Eq. (A24). From the definition of $\mathbf{F}$ and $\mathbf{v}$ and Eq. (A17), for $z<-h$, $\mathbf{F}$ is given by

$$\begin{aligned}\mathbf{F}(z|\mathbf{K},\mathbf{k},\Omega,0)=&\left\langle\int\frac{d\mathbf{Q}_1 d\mathbf{Q}_2}{(2\pi)^2}e^{-i\mathbf{Q}_2\cdot\mathbf{R}}e^*(z,\mathbf{Q}_2)\alpha^*(\mathbf{Q}_2|\mathbf{K}-\mathbf{k}/2)/[2i\beta_1^*(\mathbf{K}-\mathbf{k}/2)]\right.\\&\left.\times\frac{\mathbf{Q}_1}{\rho(z)\Omega}e^{i\mathbf{Q}_1\cdot\mathbf{R}}e(z,\mathbf{Q}_1)\alpha(\mathbf{Q}_1|\mathbf{K}-\mathbf{k}/2)/[-2i\beta_1(\mathbf{K}+\mathbf{k}/2)]\right\rangle\bigg|_{\mathbf{R}=0}\\&=\int\frac{d\mathbf{Q}}{(2\pi)^2}e^*(z,\mathbf{Q}-\mathbf{k}/2)e(z,\mathbf{Q}+\mathbf{k}/2)\Gamma_{\mathbf{Q},\mathbf{K}}(\mathbf{k})\frac{\mathbf{Q}+\mathbf{k}/2}{\rho(z)\Omega 4\beta_1(\mathbf{K}+\mathbf{k}/2)\beta_1^*(\mathbf{K}-\mathbf{k}/2)},\end{aligned} \tag{A48}$$

where sum and difference variables $\mathbf{Q}=(\mathbf{Q}_1+\mathbf{Q}_2)/2$ and $\mathbf{q}=\mathbf{Q}_1-\mathbf{Q}_2$ have been substituted for $\mathbf{Q}_1$ and $\mathbf{Q}_2$. It is straightforward to show that

$$\Gamma_{\mathbf{Q},\mathbf{P}}(-\mathbf{q})^*=\Gamma_{\mathbf{Q},\mathbf{P}}(\mathbf{q}), \tag{A49}$$

and then that

$$\begin{aligned}&\Omega\mathbf{q}\cdot[\mathbf{F}(z|\mathbf{P},\mathbf{q},\Omega,0)+\mathbf{F}^*(z|\mathbf{P},-\mathbf{q},\Omega,0)]\\&\quad=\int\frac{d\mathbf{Q}}{(2\pi)^2}e(z,\mathbf{Q}+\mathbf{q}/2)e(z,\mathbf{Q}-\mathbf{q}/2)2\mathbf{Q}\cdot\mathbf{q}\Gamma_{\mathbf{Q},\mathbf{P}}(\mathbf{q})/[4\rho(z)\beta_1(\mathbf{P}+\mathbf{q}/2)\beta_1(\mathbf{P}-\mathbf{q}/2)^*].\end{aligned} \tag{A50}$$

With this result, the contribution A_1^F of the terms in S which involve $\mathbf{F}$ in the region $-H<z<-h$ to A_1 becomes

$$A_1^F=\lim_{q\to 0}\frac{-1}{N}\int_{-H}^{-h}dz\int d\mathbf{p}\int d\mathbf{Q}\frac{\rho e(z,\mathbf{Q}+\mathbf{q}/2)e(z,\mathbf{Q}-\mathbf{q}/2)^*\mathbf{q}\cdot\mathbf{Q}}{iq^2\rho(z)}\Gamma_{\mathbf{Q},\mathbf{P}}(\mathbf{q})\frac{\lambda^0(\mathbf{q})}{4\beta_1(\mathbf{P}+\mathbf{q}/2)\beta_1(\mathbf{P}-\mathbf{q}/2)^*\Delta G_{\mathbf{P}}^0(\mathbf{q})}\phi_{\mathbf{P}}^0(\mathbf{q}). \tag{A51}$$

To obtain to an expression for A_1^F which is analogous to the result for A_1^Y in that λ^0 is eliminated, we again use Eq. (A35) and the eigenfunction expansion of Φ, but now for nonzero $\mathbf{q}$:

$$A_1^F=\lim_{q\to 0}\frac{-1}{N}\int_{-H}^{-h}dz\int d\mathbf{P}\int d\mathbf{Q}$$
$$\times\frac{\rho e(z,\mathbf{Q}+\mathbf{q}/2)\gamma(\mathbf{Q}+\mathbf{q}/2)e(z,\mathbf{Q}-\mathbf{q}/2)^*\gamma(\mathbf{Q}-\mathbf{q}/2)^*\mathbf{q}\cdot\mathbf{Q}}{iq^2\rho(z)}$$
$$\times[1/V(\mathbf{Q}+\mathbf{q})V^*(\mathbf{Q}-\mathbf{q})]\langle T_{1,2}T_{1,2}^*\rangle^{-1}_{\mathbf{Q},\mathbf{P}}(\mathbf{q})\phi^0_{\mathbf{P}}(\mathbf{q}). \tag{A52}$$

Now, however, it is awkward to take the $q\to 0$ limit of $ee^*\gamma\gamma^*/(VV^*)]\langle T_{1,2}T_{1,2}^*\rangle^{-1}\phi^0(\mathbf{q})$. Instead, we simply write

$$\lim_{q\to 0}\int e(z,\mathbf{Q}+\mathbf{q})e^*(z,\mathbf{Q}-\mathbf{q})\gamma\gamma^*\frac{1}{VV^*}$$
$$\times\langle T_{1,2}T_{1,2}^*\rangle^{-1}_{\mathbf{Q},\mathbf{P}}(\mathbf{q})\phi^0_{\mathbf{P}}(\mathbf{q})d\mathbf{P}$$
$$=+i\mathbf{q}\cdot\mathbf{Q}B_Q^{int}(z)\phi^0_{\mathbf{Q}}(0)+O(q^2) \tag{A53}$$

for small q. In this way A_1 is shown to be given by

$$A_1=A_1^Y+A_1^F$$
$$=\Lambda_0\Bigg\{-1+1/(N\Lambda_0)$$
$$\times\int d\mathbf{p}\int_0^\infty dz(p^2/2)e^{-2\,\mathrm{Im}\,\beta_1(\mathbf{p})z}B_p\phi^0_{\mathbf{p}}$$
$$+1/(N\Lambda_0)\int d\mathbf{Q}\int_{-H}^{-h}dz(Q^2/2)\frac{\rho}{\rho(z)}B_Q^{int}(z)\phi^0_{\mathbf{Q}}\Bigg\}$$
$$+\frac{-1}{2iN}\lim_{q\to 0}\int d\mathbf{p}(2\pi)^2\rho_1\int_{-h}^{0}dz[\mathbf{F}(z|\Omega,0,\mathbf{p},\mathbf{q}/2)$$
$$+\mathbf{F}^*(z|\Omega,0,\mathbf{p},-\mathbf{q}/2)]\cdot\mathbf{q}\frac{\lambda(\mathbf{q},0)}{q^2\Delta G_{\mathbf{p}}(\mathbf{q})}\phi^0_{\mathbf{p}}(\mathbf{q}). \tag{A54}$$

If the selvage region is small, it may be possible to neglect the last integral. Alternatively, it may be just as well to extend the last integral to the region $-H<z<0$, and drop the term involving the functions $B_Q^{int}(z)$.

APPENDIX B

The purpose of this appendix is to establish Eq. (64) of the text. The main ingredient in the demonstration is the symmetry expressed by Eq. (39). For small $\mathbf{q}$ and $\omega=0$ this symmetry implies

$$\delta\left(\frac{1}{\Delta G_{\mathbf{p}}(\mathbf{q})}\right)H^0_{\mathbf{p},\mathbf{p}'}+\frac{1}{\Delta G_{\mathbf{p}}(0)}H^1_{\mathbf{p},\mathbf{p}'}(\mathbf{q})$$
$$=\delta\left(\frac{1}{\Delta G_{-\mathbf{p}'}(-\mathbf{q})}\right)H^0_{\mathbf{p}',-\mathbf{p}}+\frac{1}{\Delta G_{-\mathbf{p}'}(0)}H^1_{-\mathbf{p}',\mathbf{p}}(-\mathbf{q}), \tag{B1}$$

where $\delta[1/G_{\mathbf{p}}(\mathbf{q})]$ is the first-order (in $\mathbf{q}$) variation of $1/G_{\mathbf{p}}(\mathbf{q})$. Note that $G_{\mathbf{p}}(\mathbf{q})=G_{-\mathbf{p}}(-\mathbf{q})$ and that $\phi^0_{\mathbf{p}}=\phi^0_{-\mathbf{p}}$, but that, in general, $\phi^m_{\mathbf{p}}\neq\phi^m_{-\mathbf{p}}$ for $m\neq 0$. These symmetries mean that

$$\left\langle\phi^m_{-\mathbf{p}}\frac{1}{\Delta G^0_{\mathbf{p}}}H^1_{\mathbf{p},\mathbf{p}'}(\mathbf{q})\phi^0_{\mathbf{p}'}\right\rangle$$
$$=\left\langle\phi^m_{-\mathbf{p}}\frac{1}{\Delta G^0_{-\mathbf{p}'}}H^1_{-\mathbf{p}',-\mathbf{p}}(-\mathbf{q})\phi^0_{\mathbf{p}'}\right\rangle$$
$$-\left\langle\phi^m_{-\mathbf{p}}\delta\left(\frac{1}{\Delta G_{\mathbf{p}}(\mathbf{q})}\right)H^0_{\mathbf{p},\mathbf{p}'}\phi^0_{\mathbf{p}'}\right\rangle$$
$$+\left\langle\phi^m_{-\mathbf{p}}\delta\left(\frac{1}{\Delta G_{-\mathbf{p}'}(-\mathbf{q})}\right)H^0_{-\mathbf{p}',-\mathbf{p}}\phi^0_{\mathbf{p}'}\right\rangle. \tag{B2}$$

The first term on the right can be rewritten as

$$\left\langle\phi^0_{-\mathbf{p}'}\frac{1}{\Delta G^0_{-\mathbf{p}'}}H^1_{-\mathbf{p}',-\mathbf{p}}(-\mathbf{q})\phi^m_{-\mathbf{p}}\right\rangle$$
$$=\left\langle\phi^0_{\mathbf{p}'}\frac{1}{\Delta G^0_{\mathbf{p}'}}H^1_{\mathbf{p}',\mathbf{p}}(-\mathbf{q})\phi^m_{\mathbf{p}}\right\rangle \tag{B3}$$

because of the inversion symmetry of ϕ^0, and because the various $\mathbf{p}$'s are really just dummy variables of integration. The third term can be manipulated similarly, and when $H^0\phi^m=\lambda^m\phi^m$ is used, the result is

$$\left\langle\phi^m_{-\mathbf{p}}\frac{1}{\Delta G^0_{\mathbf{p}}}H^1_{\mathbf{p},\mathbf{p}'}(\mathbf{q})\phi^0_{\mathbf{p}'}\right\rangle$$
$$=\left\langle\phi^0_{\mathbf{p}'}\frac{1}{\Delta G^0_{\mathbf{p}'}}H^1_{\mathbf{p}',\mathbf{p}}(-\mathbf{q})\phi^m_{\mathbf{p}}\right\rangle$$
$$+(\lambda^m-\lambda^0)\left\langle\phi^m_{-\mathbf{p}}\delta\left(\frac{1}{\Delta G_{\mathbf{p}}(\mathbf{q})}\right)\phi^0_{\mathbf{p}}\right\rangle. \tag{B4}$$

In the usual scheme (quantum mechanics), the weight function ΔG is replaced by 1, and there is no equivalent of the last term which involves the variation of ΔG with $\mathbf{q}$. In that case, equivalence of the first two terms expresses the self-adjoint character of H. It is the variation of ΔG that changes matters here.

Equation (64) of the text follows from Eq. (B4) and the definition of the regular part of Φ,

$$\Phi^{reg}_{\mathbf{p},\mathbf{p}'}=\sum_{m\neq 0}\frac{\phi^m_{\mathbf{p}}\phi^m_{-\mathbf{p}'}}{\lambda^m-\lambda^0}. \tag{B5}$$

Let

$$J=\left\langle \phi^0\frac{1}{\Delta G^0}A(\mathbf{q})\Phi^{req}\frac{1}{\Delta G^0}H^1(\mathbf{q})\phi^0\right\rangle, \tag{B6}$$

where A is any operator diagonal in $\mathbf{p}$, e.g.,

$$A_{\mathbf{p},\mathbf{p}'}(\mathbf{q})=A_{\mathbf{p}}(\mathbf{q})\delta_{\mathbf{p},\mathbf{p}'}. \tag{B7}$$

For the purposes of this paper we use

$$A_{\mathbf{p}}(\mathbf{q})=i\mathbf{q}\cdot\mathbf{p}\Lambda(\mathbf{p},\Omega). \tag{B8}$$

Use Eq. (B5) to replace Φ^{reg} in Eqs. (B6) and (B4) to replace $\langle\phi^m(1/\Delta G^0)H^1\phi^0\rangle$. Then note that in the sum over states which no longer contains $\lambda^m-\lambda^0$ the $m=0$ term can be added freely since in fact isotropy causes it to vanish. Applying the completeness relation

$$\sum_m \phi^m_{\mathbf{p}}\phi^m_{-\mathbf{p}}=\Delta G^0\delta_{\mathbf{p},\mathbf{p}'} \tag{B9}$$

then gives the desired result

$$J=\left\langle \phi^0_{-\mathbf{p}}\frac{1}{\Delta G^0_{-\mathbf{p}}}A_{\mathbf{p}}(\mathbf{q})\Delta G^0_{\mathbf{p}}\delta\left(\frac{1}{\Delta G_{\mathbf{p}}(\mathbf{q})}\right)\phi^0_{\mathbf{p}}\right\rangle + \left\langle \phi^0_{-\mathbf{p}'}\frac{1}{\Delta G^0_{\mathbf{p}'}}H^1_{\mathbf{p}',\mathbf{p}}(-\mathbf{q})\Phi^{reg}_{\mathbf{p}',\mathbf{p}''}\frac{1}{\Delta G^0_{\mathbf{p}''}}A_{\mathbf{p}''}(-\mathbf{q})\phi^0_{\mathbf{p}''}\right\rangle. \tag{B10}$$

Isotropy dictates that $\Phi^{reg}_{\mathbf{p},\mathbf{p}'}$ depends only on $|\mathbf{p}|,|\mathbf{p}'|$ and $\mathbf{p}\cdot\mathbf{p}'$. The appearance of $A_{\mathbf{p}''}(-\mathbf{q})$ results from

$$\left\langle \phi^0_{\mathbf{p}}\frac{1}{\Delta G^0_{\mathbf{p}}}A_{\mathbf{p}}(\mathbf{q})\phi^m_{\mathbf{p}}\right\rangle=\left\langle \phi^m_{-\mathbf{p}}\frac{1}{\Delta G^0_{-\mathbf{p}}}A_{-\mathbf{p}}(\mathbf{q})\phi^0_{-\mathbf{p}}\right\rangle =\left\langle \phi^m_{-\mathbf{p}}\frac{1}{\Delta G^0_{\mathbf{p}}}A_{\mathbf{p}}(-\mathbf{q})\phi^0_{\mathbf{p}}\right\rangle. \tag{B11}$$

The dummy variable in $\phi^m_{\mathbf{p}}$ needs to be changed to $-\mathbf{p}$, because it is $\phi^m_{-\mathbf{p}}$ that appears in Φ^{reg}.

The first order variation of $1/\Delta G$ is given by

$$\delta\frac{1}{\Delta G(\mathbf{q})}=-\left(\frac{1}{\Delta G^0}\right)^2\delta\Delta G(\mathbf{q})=-\left(\frac{1}{\Delta G^0}\right)^2 i\mathbf{q}\cdot\mathbf{p}\frac{\partial\,\mathrm{Re}\,g}{\partial p^2}, \tag{B12}$$

so that

$$J=\left\langle \phi^0\frac{1}{\Delta G^0}A(\mathbf{q})\Phi^{reg}\frac{1}{\Delta G^0}H^1(\mathbf{q})\phi^0\right\rangle =-\left\langle \phi^0_{-\mathbf{p}}\frac{1}{\Delta G^0_{-\mathbf{p}}}A_{\mathbf{p}}(\mathbf{q})i\mathbf{q}\cdot\mathbf{p}\frac{\partial\,\mathrm{Re}\,g}{\partial p^2}\frac{1}{\Delta G^0_{\mathbf{p}}}\phi^0_{\mathbf{p}}\right\rangle + \left\langle \phi^0_{-\mathbf{p}'}\frac{1}{\Delta G^0_{\mathbf{p}'}}H^1_{\mathbf{p}',\mathbf{p}}(-\mathbf{q})\Phi^{reg}_{\mathbf{p}',\mathbf{p}''}\frac{1}{\Delta G^0_{\mathbf{p}''}}A_{\mathbf{p}''}(-\mathbf{q})\phi^0_{\mathbf{p}''}\right\rangle. \tag{B13}$$

Equation (67) for $\Delta\lambda(\mathbf{q})$ can be written as

$$\Delta\lambda(\mathbf{q})=-\left\langle \phi^0\frac{1}{\Delta G^0}A\Phi^{reg}\frac{1}{\Delta G^0}H^1(\mathbf{q})\phi^0\right\rangle + \left\langle \phi^0\frac{1}{\Delta G^0}\mathbf{q}\cdot\frac{\partial U}{\partial\mathbf{q}}\Phi^{reg}\frac{1}{\Delta G^0}\left[A-\mathbf{q}\cdot\frac{\partial U}{\partial\mathbf{q}}\right]\phi^0\right\rangle + \left\langle \phi^0\frac{1}{\Delta G^0}H^2\phi^0\right\rangle. \tag{B14}$$

If identity (B13) is used in the first term, the perturbation of the eigenvalue, $\Delta\lambda(\mathbf{q})$, is shown to be given by Eq. (65) of the text.

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